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FLOER HOMOLOGY OF CONNECTED SUM OF HOMOLOGY 3-SPHERES

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0. INTRODUCTION

LET M be a homology 3-sphere and $R(M)$ be the space consisting of the conjugacy classes of $SU(2)$ representations of the fundamental group $\pi_1(M)$. In this paper we study the Floer homology group [11] $I_*(M)$ in the case when $R(M)$ is of positive dimension. Such a situation occurs naturally in the case when M is a connected sum of two homology 3-spheres.

The Floer homology group is defined by making use of Morse theory on an infinite-dimensional space (of Gauge equivalence classes of connections). The "Morse function" is the Chern–Simons functional. The set $R(M)$ is the set of critical points of this functional. The Chern–Simons functional is a Morse function if $R(M)$ is discrete and its Hessian is nondegenerate. (In our infinite-dimensional situation the latter is equivalent to the vanishing of an appropriate first cohomology group.) In that case Floer homology is given by counting "tunnel numbers" between two flat connections.

There is a generalization of Morse theory by Bott [4], etc., to the case when the critical point set is a smooth submanifold and the Hessian on the normal bundle is nondegenerate. In this paper we are concerned with an infinite-dimensional version of this case.

Floer's definition of Floer homology is based on Witten's interpretation [26] of Morse theory. Hence, our first task is to modify Witten's work to Bott's situation. In the finite-dimensional case, the analytic counterpart of Witten's work is generalized by Bismut [3] to Bott's situation. That is, Bismut proved Morse inequality using Witten's idea. But, for our purpose, we need more. Namely, we need to find the homology groups of the manifold using a Bott–Morse function. Our result is the following:

THEOREM 1.2. *Let X be a finite-dimensional manifold and f be a Morse function on it in Bott's sense. We also assume that f is weakly self-indexing. Then there exists a spectral sequence E^*_{**} such that*

$$E^*_{ij} \Rightarrow H_{i+j}(X; \mathbf{Z}) \tag{1.2.1}$$

$$E^1_{ij} \equiv H_j(R_i; \mathbf{Z}). \tag{1.2.2}$$

Here $R = \bigcup_k R_k$ is an appropriate decomposition of the critical point set R (see Section 1 for the precise assumption).

Theorem 1.2 in the finite-dimensional case might be a folk theorem in the sense that most of the specialists know how to prove it once the statement is given. But we prove it in this paper since we choose the proof which can be generalized to infinite-dimensional case directly.

Our next task is to generalize the result to the infinite-dimensional case. One can use the same topological machinery as in the finite-dimensional case. But one needs various analytic results to justify the argument.

The analysis one needs is the study of anti-self-dual connections on an open 4-manifold with product end. In Floer's and Taubes' work, one also needs to study this. The essential difference between our case and their case is that our boundary condition is degenerate. Namely, the boundary condition we put is that the anti-self-dual connection is asymptotic to a flat connection belonging to a given connected component of $R(M)$. In other words, we study the case when the boundary condition can move. This case is discussed in Mrowka's Ph.D thesis [20] (see also [25]). His result (after minor modification) can be applied to our case. So we basically follow his method but we need to add some additional results. For example, the perturbation, transversality, decay estimate, and precise arguments for compactification of moduli spaces of ASD connections. Based on these analyses we will prove:

THEOREM 2.2. *Let M satisfy Assumption* 2.1. *Then $R(M)$ is divided into $R_i(M)$, $i \in \mathbf{Z}$ and there exists a spectral sequence E^*_{**} such that*

$$E^*_{ij} \Rightarrow I_{i+j}(M) \tag{2.2.1}$$

$$E^1_{ij} \cong H_j(R_i; \mathbf{Z}). \tag{2.2.2}$$

Roughly speaking, Assumption 2.1 means that Chern–Simons functional is a Bott–Morse function and is weakly self-indexing.

We point out that this assumption is satisfied in various situation, besides the case of connected sum (which is the case we are concerned with mainly in this paper). For example, a Seifert homology sphere satisfies this assumption. The author expects that Theorem 2.2 is useful to calculate relative Donaldson polynomials for 4-manifolds with Seifert homology sphere boundaries.

The construction of the differentials in Theorem 2.2 has a significant similarity to the construction of correspondence of cycles in algebraic geometry. In fact, in some case, the relative Donaldson polynomial should be regarded as correspondence of cycles in an algebraic variety, which is a moduli space of stable bundles.

Now, let M be a connected sum of two homology spheres M_1, M_2. Let $C_*(M_1)$, $C_*(M_2)$ be the chain complexes used to define their Floer homologies. Our main result of this paper is:

THEOREM 0.0. *There exists a spectral sequence E^*_{**} such that*

$$E^*_{ij} \Rightarrow H_{i+j}(M; \mathbf{Z}) \tag{0.0.1}$$

$$E^2_{ij} \cong H_i(C_*(M_1) \otimes C_*(M_2); H_j(SO(3); \mathbf{Z})) \tag{0.0.2}$$

for $j \neq 0$.

There exists a long exact sequence

$$\begin{aligned} &\to I_i(M_1) \oplus I_i(M_2) \to E^2_{i0} \to H_i(C_*(M_1) \otimes C_*(M_2); \mathbf{Z}) \\ &\to I_{i-1}(M_1) \oplus I_{i-1}(M_2) \to E^2_{i-1\,0} \to H_{i-1}(C_*(M_1) \otimes C_*(M_2); \mathbf{Z}) \end{aligned} \tag{0.0.3}$$

The differential of the spectral sequence in Theorem 0.0 and the homomorphism $H_i(C_*(M_1) \otimes C_*(M_2); \mathbf{Z}) \to I_{i-1}(M_1) \oplus I_{i-1}(M_2)$ in (0.0.3) is described in terms of moduli spaces on M_1, M_2 only (that is without using information on the moduli space of M). The rough description of these are as follows (see Sections 4 and 5 for precise results). For simplicity, we describe the case of rational coefficient. Then the nontrivial maps in question are

$$\partial_{1,i}\colon \bigoplus_{k+\ell=i} I_k(M_1; \boldsymbol{Q}) \otimes I_\ell(M_2; \boldsymbol{Q}) \to I_{i-1}(M_1; \boldsymbol{Q}) \oplus I_{i-1}(M_2; \boldsymbol{Q}) \tag{0.1}$$

and

$$\partial_{4,i}\colon \bigoplus_j I_{i-j+4}(M_1; \boldsymbol{Q}) \otimes I_j(M_2; \boldsymbol{Q}) \to \operatorname{Ker} \partial_{1,i} \oplus \operatorname{CoKer} \partial_{1,i+1}. \tag{0.2}$$

The first map is closely related to a map $q\colon I_1(M_i) \to \boldsymbol{Z}$ defined in [9, 14]. Namely, it is given by

$$\partial_{1,i}([a] \oplus [b]) = q(a) \otimes b \pm a \otimes q(b).$$

The map $q\colon I_1(M_i) \to \boldsymbol{Z}$ is defined roughly as

$$q(a) = \#\mathcal{M}(a, \theta_0).$$

Here θ_0 is the trivial connection, and $\mathcal{M}(a, \theta_0)$ is the moduli space of ASD connections on $M \times \boldsymbol{R}$ with boundary conditions a, θ_0. The other notations are explained in Section 2. We remark that if $q\colon I_1(M_i) \to \boldsymbol{Z}$ vanishes, then Donaldson's theorem [6] can be generalized to 4-manifold which bounds M (see [9, 14] and Folk theorem 5.3 in this paper).

To describe the differential $\partial_{4,i}$ we need to introduce the monodoromy map h_i. Fix a base point p_0 on M_i and let $\ell_i = p_0 \times \boldsymbol{R}$ be a curve in $M_i \times \boldsymbol{R}$. The map

$$h_i\colon \mathcal{M}(a, b) \to SO(3)$$

is the holonomy map along this curve. The second component of the differential $\partial_{4,i}$ is given by taking a degree of the map. The first component is similar to $\partial_{1,i}$. (Roughly speaking its dual; see Sections 4 and 5 for precise result.)

The case when the coefficient is $\boldsymbol{Z}$ can be discussed in a similar way. But this case is more complicated because of the presence of $\boldsymbol{Z}_2$-torsion in the homology group of $SO(3)$.

The spectral sequence in Theorem 0.0 is obtained by applying Theorem 2.2. Let us suppose that the Chern–Simons functional is a Morse function for M_1 and M_2. Then, the Chern–Simons functional of $M_1 \# M_2$ is a Bott–Morse function and its critical submanifold is a union of components diffeomorphic either to $SO(3)$ or a point. Here the component diffeomorphic to $SO(3)$ corresponds to the connected sum $a \# b$ of irreducible connections a and b on M_1 and M_2 and the components diffeomorphic to a point corresponds to $a \# \theta$, $\theta \# b$, or $\theta \# \theta$ where θ is the trivial connection and a and b are connections on M_1 and M_2. Thus, the E_1 term of the spectral sequence is obtained.

To determine the first differentials of the spectral sequence in Theorem 0.0 we proceed in the following way. First, as we will discuss in Section 1, the spectral sequence of Theorem 2.2 is obtained by using moduli space $\mathcal{M}(R_i, R_j)$ of gradient lines with boundary value in R_i, R_j (connected component of critical submanifold) and maps $\mathcal{M}(R_i, R_j) \to R_i \times R_j$. In our case R_i, R_j is either $SO(3)$ or a point.

We construct another spectral sequence using moduli spaces $\mathcal{M}(a, a')$, $\mathcal{M}(b, b')$ of gradient lines in M_1 and M_2 and the monodromy map h_i. Namely, we consider space $(\mathcal{M}(a, a') \times \mathcal{M}(b, b)) \tilde{\times} SO(3)$ and using h_i define a map $(\mathcal{M}(a, a') \times \mathcal{M}(b, b)) \tilde{\times} SO(3) \to SO(3) \times SO(3)$. Here we regard the first factor of $SO(3) \times SO(3)$ as a critical submanifold corresponding to $a \# b$ and the second factor to the one corresponding to $a \# b'$. The construction of these spaces *a priori* is not related to the connected sum, that is the construction is done purely in terms of the moduli spaces of M_1 and M_2. Using these spaces we construct the spectral sequence and can prove that the E_2-term of this spectral sequence is given as in Theorem 0.0 and the description above.

So the final step of the proof is to show that these two spectral sequences give the same homology group. We prove it by a cobordism argument. Namely, we regard the connected sum as a 0-dimensional surgery and use the 4-manifold N which is a trace of the surgery. (The boundary of N is a disjoint union of $M_1 \# M_2$, M_1 and M_2.) We consider the moduli space of ASD connections on this space. The end of this moduli space is described by the moduli spaces of ASD connections on $M_1 \# M_2 \times \boldsymbol{R}$, $M_i \times \boldsymbol{R}$ and N. We also define a monodoromy map similar to h_i on the space of connections on N: this monodoromy map is related to h_i at the end of the moduli space. Using them we can construct chain homomorphisms between two spectral sequences (one is obtained from N and the other from $-N$). To show that these chain maps give isomorphism on homology we consider the two 4-manifolds obtained by patching N and $-N$ along $M_1 \# M_2$ or $M_1 \cup M_2$. The former is diffeomorphic to $(M_1 \times \boldsymbol{R}) \# (M_2 \times \boldsymbol{R})$. (Here $\#$ is the connected sum between two 4-manifolds; see Fig. 4.) Thus, the moduli space of ASD connections is easily described in terms of moduli spaces for $M_1 \times \boldsymbol{R}$ and $M_2 \times \boldsymbol{R}$. If we patch N and $-N$ along $M_1 \cup M_2$ and perform one-dimensional surgery along an appropriate loop ℓ, we obtain $M_1 \# M_2 \times \boldsymbol{R}$ (see Fig. 5). The holonomy map on the moduli spaces of connections on N and $-N$ are related to the holonomy along the loop ℓ. Thus, we need to know the behavior of the moduli space of ASD connections when one-dimensional surgery is done. Roughly speaking, the answer is that the moduli space of ASD connections after surgery is diffeomorphic to the set of the moduli space of ASD connections on the space before surgery whose holonomy along ℓ vanishes. Using these descriptions one can prove that the two spectral sequences coincide with each other.

Using Kronheimer's unpublished result we can calculate the boundary operator over rational coefficient in case M_i is the Poincaré homology sphere. (There are two cases according to the orientations of M_i; see Section 5 for a statement of the result.) In particular, in the case when the orientations of two Poincaré homology 3-spheres do not coincide with each other, we prove that the Floer homology of their connected sum over $\boldsymbol{Q}$ vanishes. We also point out that Theorem 0.0 may give examples of homology 3-spheres which have a 2-torsion in Floer homology. The author does not know an example with odd torsion.

The organization of this paper is as follows. The paper is divided into two parts. In Part I we give topological or geometric arguments assuming some lemmas. The proof of our results is completed there modulo analysis. Those analytic lemmas are proved in Part II. Sections 1 and 2 are devoted to the proofs of Theorems 1.2 and 2.2, respectively. Sections 3 and 4 are devoted to the proof of Theorem 0.0. In Section 3, we construct the spectral sequence. In Section 4 it is proved that the spectral sequence converges to the Floer homology of the connected sum. In Section 5 we calculate the case of connected sum of Poincaré homology spheres. Also we demonstrate explicit description of the differentials over rational coefficient. Sections 6–9 are devoted to the proof of lemma from analysis. The argument is basically the combination of various techniques developed already in Gauge theory. We call, however, the attention of the reader to Lemma 4.5, which is proved in Section 9. This lemma describes a behavior of the moduli space under 1-dimensional surgery of 4-manifold, and might be of independent interest.

Since a part of the ideas of this paper came from Prof. A. Floer, the author would like to record here which part belongs to Floer. The author had a short discussion with Floer in August 1990 (just after ICM Kyoto). At that time, answering to the author's question, Floer mentioned an exact sequence which gives the Floer homology of the connected sum. The boundary map of the exact sequence is the map (0.2). The author's impression at that time was that Floer basically knew the answer and would complete the proof and publish it in the near future. But a subsequent tragedy made it impossible for him to do so. Therefore, after that the author tried to remember what Floer told him and to complete the proof (and the statement). Then the author found that one also needs another boundary map (0.1) and spectral sequence. He also worked out the necessary analytic tools used in the proof. The author does not know how much Floer knew on the topic of this paper.

The construction of the spectral sequence in this paper has significant resemblance to a construction used in [2] to construct equivariant Floer theory. The author is inspired by it. (Their discussion uses rational coefficients, but the author feels that by combining the technique of this paper, one can argue over the integers for their purpose also.)

After the first version of this paper was completed the author received a preprint [18] which studies similar problem as our Theorem 0.0. Li's method is quite different from ours. Our Theorem 0.0, combined with Taubes' result [25], implies in particular a connected sum formula for Casson invariant as already shown by Nakamura [21].

PART I: TOPOLOGY

1. THE CASE OF FINITE-DIMENSIONAL MANIFOLD

Let X be an oriented closed manifold of finite dimension, and $f: X \to \boldsymbol{R}$ be a function on it. We put

$$R(X, f) = \{x \in X \mid df(x) = 0\}.$$

We omit X, f and write R, so that no confusion can occur. In this section, we work under the following assumptions.

Assumptions 1.1.

(1.1.1) $R(f, X)$ is smooth submanifold.

(1.1.2) The restriction of the Hessian, $\mathrm{Hess}_x f$, to the normal bundle $N_x R(X, f)$ is nondegenerate for each $x \in R(X, f)$.

(1.1.3) The Morse function f is weakly self-indexing.

The property f is weakly self-indexing will be defined later (this property is due to Austin–Braam [1]).

For $x \in R$ we denote by $\mu(x)$ the number of negative eigenvalues of $\mathrm{Hess}_x f$. By Assumption 1.1.2 this number is constant on each connected component of R. We put

$$R_k = \{x \in \boldsymbol{R} \mid \mu(x) = k\}.$$

R_k is a union of connected components of R. In this section we prove the following.

THEOREM 1.2. *Let X, f, satisfy Assumption* 1.1. *Then there exists a spectral sequence E^*_{**} such that*

$$E^*_{ij} \Rightarrow H_{i+j}(X; \boldsymbol{Z}) \tag{1.2.1}$$

$$E^1_{ij} \cong H_j(R_i; \boldsymbol{Z}). \tag{1.2.2}$$

Example 1.3. Suppose that R is discrete. In this case, E^1_{ij} is trivial unless $j = 0$. Hence, the spectral sequence is degenerate at E^2-level and reduces to a chain complex (C_k, ∂) where C_k is the free abelian group on the base of elements of R_k. This is the chain complex discussed by Witten [26].

Example 1.4. Let $\pi: X \to Y$ be a fiber bundle. Choose a Morse function $\bar{f}$ on Y and put $f = \bar{f} \circ \pi$. This function f satisfies our Assumption 1.1. Hence, we have a spectral sequence converging to $H_*(X; \boldsymbol{Z})$. In this case, we can calculate the E^2-term of this spectral sequence and find that this spectral sequence coincides with Leray–Serre spectral sequence.

Proof of Theorem 1.2. To prove Theorem 1.2, we choose a Riemannian metric g on X and put

$$\chi = -\operatorname{grad} f.$$

We consider the differential equation

$$\frac{D\ell}{dt} = \chi(\ell(t)) \tag{1.5}$$

and define

$$\tilde{\mathcal{M}}(j,k) = \left\{ \ell : \boldsymbol{R} \to X \;\middle|\; \begin{array}{l} \ell \text{ satisfies (1.5)} \\ \lim_{t \to -\infty} \ell(t) \text{ converges to an element of } R_j \\ \lim_{t \to +\infty} \ell(t) \text{ converges to an element of } R_k. \end{array} \right\}$$

On $\tilde{\mathcal{M}}(j,k)$, the group $\boldsymbol{R}$ acts by $\ell(t) \mapsto \ell(t + t_0)$. By $\mathcal{M}(j,k)$ we denote the quotient of $\tilde{\mathcal{M}}(j,k)$ by this action.

Let ϕ_t be the one-parameter group of transformations associated with χ. We remark that

$$\tilde{\mathcal{M}}(j,k) = \left\{ x \in X \;\middle|\; \begin{array}{l} \lim_{t \to -\infty} \phi_t(t) \text{ converges to an element of } R_j \\ \lim_{t \to +\infty} \phi_t(t) \text{ converges to an element of } R_k. \end{array} \right\}$$

and

$$\mathcal{M}(j,k) = \tilde{\mathcal{M}}(j,k)/\sim$$

where

$$x \sim y \iff \exists t\, \phi_t(x) = y.$$

Let $\tilde{\mathcal{M}}(\infty, k)$ be the union of all $\tilde{\mathcal{M}}(j,k)$, $j \in \boldsymbol{Z}$. We define $\mathcal{M}(j, -\infty)$ in a similar way (we regard them as subsets of X). We choose a sufficiently small neighborhood of R and let U be the complement of this set. Let $C^\infty(U)$ to be set of all smooth functions with compact support on U. We put the C^∞-topology on it. In the next lemma we write $\tilde{\mathcal{M}}(j,k)(f)$, etc., to show explicitly that they are associated to f.

LEMMA 1.6. *There exists a residual subset of $C^\infty(U)$ such that if ϕ is contained in this subset then the two submanifolds $\tilde{\mathcal{M}}(\infty, k)(f + \phi)$ and $\tilde{\mathcal{M}}(k, -\infty)(f + \phi)$ are transversal to each other.*

From now on we say that f is generic if f is replaced by $f + \phi$ where ϕ is an element of a residual subset of $C^\infty(U)$.

In case when R is discrete, this lemma is a well-known result, that is for a generic function, the gradient flow is Morse–Smale (see, for example, [19] for the proof in this case). The proof of Lemma 1.6 is exactly the same as that case and is omitted. Lemma 1.6 implies:

LEMMA 1.7. *For a generic function f we have*

$$\dim \tilde{\mathcal{M}}(j,k) = j - k + \dim R_j.$$

We now define the property we used in Definition 1.1.

Definition 1.8. *We say that f is weakly self-indexing* if $\tilde{\mathcal{M}}(j, k)$ is empty for each j, k with $j < k$.

Remark 1.9. Since the dimension of the connected components of the submanifold R_i may be different to each other, we should state Lemma 1.7 as follows. For each connected component of R_i we have a union of connected components of $\tilde{\mathcal{M}}(j, k)$ corresponding to it. Then Lemma 1.7 holds for those components. Hereafter we omit these kinds of remarks so that no confusion can occur.

We define $\Pi^{\ell}_{j,k}: \mathcal{M}(j, k) \to R_j$, $\Pi^{r}_{j,k}: \mathcal{M}(j, k) \to R_k$ by

$$\Pi^{\ell}_{j,k}([\ell]) = \lim_{t \to -\infty} \ell(t)$$

$$\Pi^{r}_{j,k}([\ell]) = \lim_{t \to +\infty} \ell(t).$$

For generic f these maps are smooth.

LEMMA 1.10. *For generic f, the two maps* $\Pi^{r}_{j,k}$ *and* $\Pi^{\ell}_{j,k}$ *are transversal to each other.*

We omit the proof of this lemma since we will give a proof of the infinite-dimensional version in Part II and since, in the finite-dimensional case, the proof of the lemma is nothing more than a standard argument.

Next we put

$$\mathcal{CM}(j, k) = \mathcal{M}(j, k) \cup \bigcup_{j > i_1 > \cdots > i_n > k} \mathcal{M}(j, i_1) \times_{R_{i_1}} \cdots \times_{R_{i_n}} \mathcal{M}(i_n, k).$$

Here we put

$$\mathcal{M}(j, i_1) \times_{R_{i_1}} \mathcal{M}(i_1, k) = \left\{(x, y) \in \mathcal{M}(j, i_1) \times \mathcal{M}(i_1, k) \mid \prod_{j, i_1}^{r}(x) = \prod_{i_1, j}^{\ell}(y)\right\}.$$

In a way similar to Lemma 1.10, we can prove that the two maps

$$\prod_{j, i_1, \ldots, i_m}^{r} : \mathcal{M}(j, i_1) \times_{R_{i_1}} \cdots \times_{R_{i_{m-1}}} \mathcal{M}(i_{m-1}, i_m) \to R_{i_m}$$

and

$$\prod_{i_m, \ldots, i_n, k}^{\ell} : \mathcal{M}(i_m, i_{m+1}) \times_{R_{i_{m+1}}} \cdots \times_{R_{i_n}} \mathcal{M}(i_n, k) \to R_{i_m}$$

are transversal to each other, for generic f.

LEMMA 1.11. *For generic f, the set* $\mathcal{CM}(j, k)$ *has a structure of compact smooth manifold with corners such that*

$$\operatorname{Int} \mathcal{CM}(j, k) = \mathcal{M}(j, k) \tag{1.11.1}$$

$$\partial \mathcal{CM}(j, k) = \bigcup_{j > i > k} \mathcal{M}(j, i) \times_{R_i} \mathcal{M}(i, k). \tag{1.11.2}$$

We omit the proof of Lemma 1.11 for the same reason as Lemma 1.10.

Now we explain the idea of the construction of the differentials in the spectral sequence in Theorem 1.2. *Roughly speaking*, we are to construct the maps

$$\partial_k : H_j(R_i; \boldsymbol{Z}) \to H_{j+k-1}(R_{i-k}; \boldsymbol{Z}).$$

One might be able to define this map by

$$\partial_k = \Big(\prod_{i,i-k}^{\mathrm{r}}\Big)_* \circ PD \circ \Big(\prod_{i,i-k}^{\ell}\Big)^* \circ PD. \tag{1.12}$$

Here by PD we denote the Poincaré dual. In chain level this map is represented by

$$u \mapsto \prod_{i,j+1-k}^{\mathrm{r}} \circ \Big(\prod_{i,j+1-k}^{\ell}\Big)^{-1}(u). \tag{1.13}$$

Using Lemma 1.7, the reader can verify easily that this map sends chains to one of the appropriate dimensions in case suitable transversality condition is satisfied for u. But in fact, formula (1.12) does not make sense since the space $\mathscr{CM}(j, k)$ has a (codimension one) boundary. This is quite natural since the differential d_k cannot be defined on $H_j(R_i; \boldsymbol{Z})$, but only on its subquotient. Thus, we have to perform the construction in chain level and justify formula (1.12). For this purpose we need some notations.

Definition 1.14 (cf. Gromov [15]). A finite simplicial complex P is said to be an *abstract geometric chain* of dimension n if:

(1.14.1) There is a subspace P_{reg} which is an oriented manifold of dimension n with boundary, $\partial P_{\mathrm{reg}}$.

(1.14.2) We put $P_{\mathrm{s}} = P - P_{\mathrm{reg}}$. Then P_{s} is a subcomplex of P.

(1.14.3) $\dim P_{\mathrm{s}} \leq n - 2$.

(1.14.4) We put $\partial P = \overline{\partial P_{\mathrm{reg}}}$. Then, $\dim P_{\mathrm{s}} \cap \partial P \leq n - 3$.

For a smooth manifold X, a pair (P, f) of an abstract geometric chain P and a continuous map $f: P \to X$ is said to be a *geometric chain* of X if f is piecewise smooth and the restrictions of f to P_{reg} and $\partial P_{\mathrm{reg}}$ are smooth.

Let $C_n^g(X)$ be the set of all geometric chains on X. We consider a free abelian group over a basis of all geometric chains. We divide it by the relation

$$(P, f) + (P', f') = (P \cup P', f \cup f')$$
$$-(P, f) = (-P, f).$$

Let $C_n^g(X)$ be the quotient. Here $P \cup P'$ is the disjoint union and $-P$ is the same complex with the opposite orientation on P_{reg}. We define $\partial: C_n^g(X) \to C_{n-1}^g(X)$ by $\partial(P, f) = (\partial P, f)$. It is easy to see $\partial\partial = 0$.

Now we remark that if $\partial(P, f) = 0$, then its fundamental class in $H_n(X; \boldsymbol{R})$ is well defined. Therefore, the proof of the following lemma is fairly standard.

Lemma 1.15.

$$H_*(C_n^g(X); \partial) \cong H_*(X; \boldsymbol{Z}).$$

Next we remark that formula (1.13) makes sense only if the chain u there satisfies some transversality conditions. So we will consider "generic" geometric chains. Let $Y_1, \ldots, Y_q$ be a family of smooth manifolds and $g_i: Y_i \to X$ be smooth maps.

Definition 1.16. A geometric chain (P, f) of X is said to be transversal to $((Y_1, g_1), \ldots, (Y_q, g_q))$ if

(1.16.1) the restriction of f to P_{reg} is transversal to g_i,

(1.16.2) the restriction of f to $\partial P_{\mathrm{reg}}$ is transversal to g_i,

(1.16.3) the restriction of f to each simplex of P is transversal to g_i.

Let $C_n^{gt}(X)$ be the set of all transversal geometric n chains of X. Note that $C_n^{gt}(X)$ is a subcomplex of $C_n^{g}(X)$.

LEMMA 1.17.

$$H_*(C_n^{gt}(X); \partial) \cong H_*(X; \mathbf{Z}).$$

For the proof we first remark the following.

SUBLEMMA 1.18. *Let N be a manifold, U an open subset and $f: N \to X$ a smooth map. Suppose that $N - U$ is compact and that the restriction of f to U is transversal to g_i. Then for arbitrary ε_k there exists f_ε such that*

(1.18.1) $f = f_\varepsilon$ *on* U,

(1.18.2) f *is transversal to* g_i,

(1.18.3) $|f - f_\varepsilon|_{C^k} < \varepsilon_k$.

The proof is a standard transversality argument and is omitted. We remark here that it is unnecessary to perturb g_i to make them transversal to f. In other words, it is enough to perturb one of the maps.

Using Sublemma 1.18, we can prove Lemma 1.17 by perturbing the map f inductively. Since the argument is a straightforward generalization of the proof of the usual transversality theorem, we omit the details.

Now we are in the position to construct the maps of formula (1.13). We put $X = R_j$ and take $\mathcal{M}(j, j-k), \ldots, \mathcal{M}(j, i_1) \times_{R_{i_1}} \cdots \times_{R_{i_n}} \mathcal{M}(i_n, j-k)$ as Y_i. The maps g_i are $\Pi^{\mathfrak{r}}_{***}$. Let $(P, f) \in C_n^{gt}(R_j)$. Then the fiber product

$$P' = P \times_{R_j} \mathcal{M}(j, j-k)$$

is an abstract geometric chain of dimension $n + k - 1$. There is an induced map $P' \to \mathcal{M}(j, j-k)$. Let $f': P' \to R_{j-k}$ be the composition of this map and $\Pi^{\mathfrak{r}}_{j,j-k}$. It is easy to see that

$$(P', f') \in C^{gt}_{n+k-1}(R_{j-k}).$$

We define

$$\partial_k : C_n^{gt}(R_j) \to C^{gt}_{n+k-1}(R_{j-k})$$

by $\partial_k(P, f) = (P', f')$. We define

$$\partial_0 = (-1)^{n+j}\partial,$$

where ∂ is the usual boundary operator. Up to sign the following is an immediate consequence of Lemma 1.11.

LEMMA 1.19.

$$\sum_{m=0}^{n} \partial_m \partial_{n-m} = 0.$$

Proof. First we fix our sign convention. For an abstract geometric chain P, we define an orientation of ∂P such that

$$\boldsymbol{R} \oplus T_x(\partial P) \cong T_x(P)$$

is an isomorphism of oriented vector spaces. Let $Y_1 \to R$, $Y_2 \to R$ be two smooth maps transversal to each other. Assume that Y_i and R are oriented. Then we define an orientation on the fiber product $Y_1 \times_R Y_2$ such that

$$(-1)^{\dim R \cdot \dim Y_2} T_{(x,y)}(Y_1 \times_R Y_2) \oplus T_r R \cong T_x Y_1 \oplus T_y Y_2$$

is an isomorphism of oriented vector spaces. The factor $(-1)^{\dim R \cdot \dim Y_2}$ may look strange at first sight. But it is easy to see that this factor is necessary to make the diffeomorphism

$$(Y_1 \times_{R_1} Y_2) \times_{R_2} Y_3 \cong Y_1 \times_{R_1} (Y_2 \times_{R_2} Y_3)$$

compatible with orientation. Combining these two conventions we define orientations on $\mathcal{M}(i,j)$ as follows. For simplicity, we assume that R_0 is one point and that $\mathcal{M}(R_0, j)$ is nonempty for each j. (The technique one uses to remove them is the same as in the case of [11] and is omitted.) We choose an (arbitrary) orientation on each $\mathcal{M}(R_0, j)$ and on R_j. Then we define an orientation on $\mathcal{M}(i,j)$ such that

$$\boldsymbol{R} \oplus T_x\mathcal{M}(R_0, i) \oplus T_y\mathcal{M}(i,j) \oplus T_r R_i \cong (-1)^{\dim R_i \cdot (i-j)} T_z\mathcal{M}(R_0, j)$$

is an isomorphism of oriented vector spaces. (Note that the sign factor on the right hand side is equal to $(-1)^{\dim R_i(\dim R_i + i - j - 1)}$.) Now we verify the following.

Sublemma 1.20.

$$\partial\mathcal{M}(R_i, R_j) = (-1)^{i + \dim R_i} \bigcup_k \mathcal{M}(i,k) \times_{R_k} \mathcal{M}(k,j).$$

Proof. Using Lemma 1.11 we have to only verify that the orientations are consistent. Let ε be the sign such that

$$\partial\mathcal{M}(R_i, R_j) = \varepsilon \bigcup_k \mathcal{M}(i,k) \times_{R_k} \mathcal{M}(k,j).$$

Let $\boldsymbol{R}_1 = \boldsymbol{R}_2 = \boldsymbol{R}$. Then by our sign convention, we have

$$\begin{aligned} &\boldsymbol{R}_1 \oplus \boldsymbol{R}_2 \oplus T\mathcal{M}(R_0, i) \oplus T\mathcal{M}(i,k) \oplus T\mathcal{M}(k,j) \oplus TR_i \oplus TR_k \\ &\cong (-1)^{\dim R_i(i-k) + (k+j+1+\dim R_k)(\dim R_i)} \boldsymbol{R}_1 \oplus T\mathcal{M}(R_0, k) \oplus T\mathcal{M}(k,j) \oplus TR_k \\ &\cong (-1)^{\dim R_i(i-j-1) + \dim R_k(k-j) + \dim R_i \dim R_k} T\mathcal{M}(R_0, j). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\boldsymbol{R}_1 \oplus \boldsymbol{R}_2 \oplus T\mathcal{M}(R_0, i) \oplus T\mathcal{M}(i,k) \oplus T\mathcal{M}(k,j) \oplus TR_i \oplus TR_k \\ &\cong \varepsilon(-1)^{i + \dim R_i \dim R_k + \dim R_k(k-j)} \boldsymbol{R}_2 \oplus T\mathcal{M}(R_0, i) \oplus T\mathcal{M}(i,j) \oplus TR_i \\ &\cong \varepsilon(-1)^{i + \dim R_i \dim R_k + (k-j)\dim R_k + \dim R_i(i-j)} T\mathcal{M}(R_0, j). \end{aligned}$$

The lemma follows by comparing these two formulas.

Sublemma 1.21. *Let P_1, P_2, be abstract geometric chains and $f_i\colon P_i \to R$ be smooth maps transversal to each other. Then we have*

$$\partial(P_1 \times_R P_2) = \partial P_1 \times_R P_2 + (-1)^{\dim P_1 + \dim R} P_1 \times_R \partial P_2.$$

Proof. The proof up to sign is obvious. One can verify the sign in a way similar to the proof of Sublemma 1.21. The proof is omitted.

Now we are in the position to complete the proof of Lemma 1.18. Let $(P, f) \in C_n^{gt}(R_i)$. Then we have

$$\begin{aligned}\partial(P \times_{R_i} \mathscr{M}(i,j)) &= \partial P \times_{R_i} \mathscr{M}(i,j) + (-1)^{n+\dim R_i} P \times_{R_i} \partial \mathscr{M}(i,j) \\ &= \partial P \times_{R_i} \mathscr{M}(i,j) + (-1)^{n+i} \sum_k P \times_{R_i} \mathscr{M}(i,k) \times_{R_k} \mathscr{M}(k,j).\end{aligned}$$

The lemma follows immediately.

Now we define a chain complex $(\hat{C}_*(X, f), \hat{\partial})$ as follows:

$$\hat{C}_*(X, f) = \bigoplus_{m \in Z} C_{n-m}^{gt}(R_m)$$

$$\hat{\partial} = \oplus \partial_m.$$

Lemma 1.19 implies that $\hat{\partial}\hat{\partial} = 0$.

Lemma 1.22. *$H_*(\hat{C}_*(X, f); \hat{\partial})$ is independent of f (satisfying Assumption 1.1).*

We omit the proof of this lemma since we will prove its infinite-dimensional version in the next section.

Lemma 1.20 implies, in particular, that $H_*(\hat{C}_*(X, f); \hat{\partial})$ for any f is isomorphic to that obtained when f is a Morse function (that is when R is discrete). In that case, it is classical that this homology group coincides with the homology of X. Thus, we have:

Lemma 1.22.

$$H_*(\hat{C}_*(X, f); \hat{\partial}) \cong H_*(X; Z).$$

Now we are in a position to complete the proof of Theorem 1.2. By the definition there is a stratification of our complex $(\hat{C}_*(X, f), \hat{\partial})$; namely, we put

$$\mathscr{F}_k(\hat{C}_n(X, f)) = \bigoplus_{m \leq k} C_{n-m}^{gt}(R_m).$$

It is easy to see that $\mathscr{F}_k(\hat{C}_n(X, f))$ is a subcomplex. Furthermore,

$$\mathscr{g}r_*(\mathscr{F}_*(\hat{C}_*(X, f))) \cong (C_*^{gt}(R_*), \partial).$$

Therefore, Theorem 1.2 is a consequence of standard homological algebra and Lemma 1.17.

2. THE CASE OF FLOER HOMOLOGY

Our main interest in this paper is not a finite-dimensional case as in the last section but is in the case when the manifold X there is replaced by the set of Gauge equivalence classes of connections on a homology 3-sphere M. First we define some notations:

$$\mathscr{A}(M) = \{d + a \mid a \in \Gamma(M; \Lambda^1 \otimes su(2))\}$$

$$\hat{\mathscr{G}}(M) = \{g : M \to SU(2) \mid C^\infty\text{-maps}\}$$

$$\mathscr{G}(M) = \{g \in \hat{\mathscr{G}}(M) \mid \deg g = 0\}$$

$$\mathscr{B}(M) = \mathscr{A}(M)/\hat{\mathscr{G}}(M)$$

$$\tilde{\mathscr{B}}(M) = \mathscr{A}(M)/\mathscr{G}(M)$$

where $\hat{\mathscr{G}}(M)$ acts on $\mathscr{A}(M)$ by

$$g^*(d+a) = g^{-1}dg + g^{-1}ag.$$

The Chern–Simons functional $cs:\tilde{\mathscr{B}}(M) \to \boldsymbol{R}$ is defined by

$$cs(a) = \int_M \mathrm{Tr}(\tfrac{1}{2}a \wedge da + \tfrac{1}{3}a \wedge a \wedge a).$$

Here and hereafter, we write a in place of $d + a$. In the theory of Floer homology [11, 24], we replace X and f in the last section by $\tilde{\mathscr{B}}(M)$ and cs, respectively. In this case, the set of critical values, R, in Section 1 corresponds to the set of flat connections. Since we divide it by Gauge group $\mathscr{G}(M)$, it is identified to $\boldsymbol{Z}$ times the set

$$\bar{R}_+(M) = \mathrm{Hom}(\pi_1(M), SU(2))/\sim.$$

Here $a \sim b$ means that they are conjugate to each other. We put $R_+(M) = \bar{R}_+(M) \times \boldsymbol{Z} \subseteq \tilde{\mathscr{B}}(M)$. For $[a] \in R_+(M)$, let $su(2)^a$ be the flat vector bundle over M associated with the composition of the representation a and the adjoint representation. In this situation, we can translate Assumption 1.1 as follows.

Assumption 2.1.
(2.1.1) $R_+(M)$ is a smooth manifold.
(2.1.2) For each $[a] \in R_+(M)$ we have

$$T_{[a]}R_+(M) \cong H^1(M; su(2)^a).$$

(2.1.3) The moduli space $\mathscr{M}_{\ell,\delta}(V, W)$ is empty if $\eta(V) < \eta(W)$ or $\eta(V) = \eta(W)$, $V \neq W$. (The notations will be defined later. This condition corresponds to Assumption (1.1.3).)

Let $\bar{R}(M)$ be $\bar{R}_+(M)$ minus the trivial connection, $R(M) \cong \bar{R}_+(M) \times \boldsymbol{Z}$, and $\tilde{R}(M)$ be its inverse image in $\mathscr{A}(M)$. The purpose of this section is to show the following infinite-dimensional version of Theorem 1.2.

THEOREM 2.2. *Let M satisfy Assumption* 2.1. *Then, $R(M)$ is divided into $R_i(M)$, $i \in \boldsymbol{Z}$ and there exists a spectral sequence E^*_{**} such that*

$$E^*_{ij} \Rightarrow I_{i+j}(X) \tag{2.2.1}$$

$$E^1_{ij} \cong H_j(R_i; \boldsymbol{Z}). \tag{2.2.2}$$

Here $I_*(M)$ stands for the Floer homology of M.

To prove Theorem 2.2, we need several analytic results, which we state in this section and will prove in Part II. Once they are established the proof goes in exactly the same way as Theorem 1.2.

To state the analytic results we will define some notation. We fix a metric on M and put a product metric on $M \times \boldsymbol{R}$. Let E be a vector bundle with fiber metric and u be a section of compact support. Using the function e_δ defined at the beginning of Section 6, we put

$$(\|u\|^p_{\ell,\delta})^p = \sum_{k \le \ell} \int_{M \times \boldsymbol{R}} e_\delta(t)|\nabla^k u|^p \, dx\, dt.$$

Let $L^2_{\ell,\delta}(M \times \boldsymbol{R}; E)$ be the completion with respect to this norm. We put

$$\Omega^0_{\ell,\delta} = L^2_{\ell,\delta}(M \times \boldsymbol{R}; su(2))$$
$$\Omega^1_{\ell,\delta} = L^2_{\ell,\delta}(M \times \boldsymbol{R}; su(2) \otimes \Lambda^1(M \times \boldsymbol{R}))$$
$$\Omega^2_{\ell,\delta} = L^2_{\ell,\delta}(M \times \boldsymbol{R}; su(2) \otimes \Lambda^2_+(M \times \boldsymbol{R})).$$

Here $\Lambda^2_+(M \times \boldsymbol{R})$ is the vector bundle of self-dual 2-forms on $M \times \boldsymbol{R}$.

An important observation by Taubes is that the set $\mathscr{M}(i,j)$ in Section 1 corresponds to the set of anti-self-dual (hereafter denoted by ASD) connections divided by appropriate group of Gauge transformations. We next define this moduli space. We put

$$\mathscr{G}(M \times \boldsymbol{R}) = \{g : M \times \boldsymbol{R} \to SU(2) \mid C^\infty\text{-maps}\}.$$

Let $g_{\pm,\pm}$, be elements of $\mathscr{G}(M \times \boldsymbol{R})$ such that $g_{\pm,\pm} \equiv \pm 1$ outside a compact subset. (We take 4 elements corresponding to the choice of combinations of $\pm$.) We put

$$\mathscr{G}_{\ell,\delta}(M \times \boldsymbol{R}) = \{g \in \mathscr{G}(M \times \boldsymbol{R}) \mid \exists u \in \Omega^0_{\ell,\delta} g \exp(-u) = g_{\pm,\pm}\}.$$

Let U_i, $i = 1, 2$ be open subsets of $R(M)$ for which there are smooth sections $s_i : U_i \to \tilde{R}(M)$. For $a \in U_1$, $b \in U_2$ we let $A_{a,b}$ be a connection on $M \times \boldsymbol{R}$ such that $A_{a,b} = s(a)$ on $M \times (-\infty, -1]$, and $A_{a,b} = s(b)$ on $M \times [1, \infty)$. Shrinking U_i if necessary, we can assume that $A_{a,b}$ depends smoothly on a, b. We put

$$\hat{\mathscr{B}}_{\ell,\delta}(U_1, U_2) = \{A \in \Gamma(M \times \boldsymbol{R}, su(2) \otimes \Lambda^1(M \times \boldsymbol{R})) \mid \exists a \in U_1\ \exists b \in U_2,\ A_{a,b} - A \in \Omega^1_{\ell,\delta}\}.$$

The group $\mathscr{G}_{\ell+1,\delta}(M \times \boldsymbol{R})$ acts on $\hat{\mathscr{B}}_{\ell,\delta}(U_1, U_2)$ by Gauge transformations. Let $\mathscr{B}_{\ell,\delta}(U_1, U_2)$ be the quotient space of this action. It is easy to see that, for $U_i' \subset U_i$, there is a canonical inclusion $\mathscr{B}_{\ell,\delta}(U_1', U_2') \to \mathscr{B}_{\ell,\delta}(U_1, U_2)$. Hence, we patch them together to obtain $\mathscr{B}_{\ell,\delta}(V, W)$ for arbitrary subsets V, W of $R(M)$. The translation along second factor of $M \times \boldsymbol{R}$ induces an $\boldsymbol{R}$-action on $\mathscr{B}_{\ell,\delta}(V, W)$. Let $\bar{\mathscr{B}}_{\ell,\delta}(V, W)$ be the quotient.

We put

$$\mathscr{M}_{\ell,\delta}(V, W) = \{[A] \in \bar{\mathscr{B}}_{\ell,\delta}(V, W) \mid *F_A = -*F_A\}.$$

Here F_A is the curvature two form and $*$ is the Hodge $*$-operator on 4-manifold $M \times \boldsymbol{R}$. Since $\mathscr{M}_{\ell,\delta}(V, W)$ is independent of δ, ℓ, we omit these suffixes so that no confusion can occur.

Next we mention perturbation. In Section 1 we perturb the gradient vector field by choosing a genetic function whose support is disjoint to a small neighborhood of R. But since our space in this section is infinite dimensional, taking an arbitrary function on $\mathscr{B}(M)$ can cause a problem. In [11], Floer uses a family of perturbations introduced by Donaldson [7]. We follow their approach. In Part II, we introduce a family of functions f_ϕ on the set of Gauge equivalent classes of connections on $M \times SU(2)$ such that the support of its members are disjoint to $\tilde{R}(M)$. Let Ψ be the Banach space parametrizing this family of perturbations. An element $\phi \in \Psi$ corresponds to a perturbation of the equation $*F_A = -*F_A$. Let $\mathscr{M}_\phi(A, B)$, etc., be the set of the solutions of the perturbed equation

$$F_A + *F_A + \operatorname{grad}_{a_t} f_\phi \wedge dt - \bar{*}\operatorname{grad}_{a_t} f_\phi = 0.$$

Here and hereafter $A = a_t + b \wedge dt$, $a_t \in \mathscr{A}(M)$, and $\bar{*}$ is a Hodge $*$-operator on M.

THEOREM 2.3. *We can divide $R(M)$ into a disjoint union R_i, $i \in \boldsymbol{Z}$, such that $\mathscr{M}_\phi(j,k)$ is a smooth manifold and*

$$\dim \mathscr{M}_\phi(j,k) = j - k + \dim R_j - 1$$

for each element ϕ of a residual subset of Ψ.

Theorem 2.3 is an analogy of Lemma 1.7 and will be proved in Part II. By definition there exist maps

$$\Pi^\ell_{j,k}: \mathscr{M}_\phi(j,k) \to R_j$$

$$\Pi^r_{j,k}: \mathscr{M}_\phi(j,k) \to R_k$$

by taking boundary values at infinity. Then the following analogue of Lemma 1.10 also holds. By [11], all of the moduli spaces $\mathscr{M}_\phi(j,k)$ are oriented.

THEOREM 2.4. *For each element ϕ in a residual subset of Ψ, the maps $\Pi^\ell_{j,k}$, $\Pi^r_{j,k}$ are smooth. Furthermore, for each i,j,k the two maps $\Pi^\ell_{j,k}$ and $\Pi^r_{i,j}$ are transversal to each other.*

We can prove also the following analogue of Lemma 1.10.

THEOREM 2.5. *For each element ϕ in a residual subset of Ψ, the set $\mathscr{C}\mathscr{M}_\phi(j,k)$ has a structure of compact smooth manifold with corners such that*

$$\operatorname{Int} \mathscr{C}\mathscr{M}_\phi(j,k) = \mathscr{M}_\phi(j,k) \tag{2.5.1}$$

$$\partial \mathscr{M}_\phi(j,k) = \bigcup_{j>i>k} \mathscr{M}_\phi(j,i) \times_{R_i} \mathscr{M}_\phi(i,k). \tag{2.5.2}$$

Using these lemmas we can discuss, in exactly the same way as in Section 1, and can construct a spectral sequence satisfying (2.2.2).

We show that this spectral sequence converges to the Floer homology of M. For this purpose we need to consider a perturbation whose support may intersect $R(M)$. It is proved in [11] ([13]) that we can choose $\phi' \in \Psi'$ such that the Conditions 2.6 below holds for

$$R'(M) = \{x \in \mathscr{B}(M) \mid \operatorname{grad}_x(cs + f_{\phi'}) = 0\}.$$

Condition 2.6.

(2.6.1) $R'(M)$ is discrete.

(2.6.2) For each $[a] \in R(M)$, the operator

$$*d_a + \operatorname{Hess}_a f_{\phi'}$$

is invertible.

We recall that the operator in (2.6.2) is Hessian of $cs + f_{\phi'}$. By the definition of Floer homology in [11], there exist $\mu: R(M) \to \boldsymbol{Z}$ and $\langle \partial a, b \rangle \in \boldsymbol{Z}$ for each $[a]$, $[b]$ with $\mu(a) = \mu(b) + 1$, such that the homology of the complex

$$C_k(M) = \bigoplus_{\mu(a)=k} \boldsymbol{Z}[a]$$

$$\partial: C_k(M) \to C_{k-1}(M), \qquad [a] \mapsto \sum \langle \partial a, b \rangle [b]$$

gives Floer homology. (For example $\langle \partial a, b\rangle$ is defined by counting the number of points of $\mathcal{M}_{\phi'}(a, b)$.) Hence, to prove Theorem 2.2, it suffices to show that following.

PROPOSITION 2.7.

$$H_*(\hat{C}_*(M); \hat{\partial}) \cong H_*(C_*(M); \partial).$$

Proof. Let ϕ_t, $t \in \boldsymbol{R}$ be a homotopy of functions on $\mathscr{B}(M)$ such that

$$\lim_{t \to -\infty} \phi_t = \phi$$

$$\lim_{t \to +\infty} \phi_t = \phi'.$$

For $a \in R(M)$, we define $\mathscr{B}(R_i, M)$ in a way similar to the definition of $\mathscr{B}_{\ell,\delta}(j, k)$. We consider the differential equation

$$F_A + *F_A + \mathrm{grad}_{a_t} f_{\phi_t} \wedge dt - \bar{*}\,\mathrm{grad}_{a_t} f_{\phi_t} = 0, \tag{2.8}$$

and put

$$\mathcal{M}(R_i, a) = \{A \in \mathscr{B}(R_i, a) \,|\, A \text{ satisfies (2.8)}\}.$$

(We remark here that we did not (and cannot) divide it by the $\boldsymbol{R}$-action, when we define $\mathcal{M}(R_i, a)$.) The following results similar to Theorems 2.3, 2.4, 2.5 will be proved in Part II and are used to prove Proposition 2.7.

LEMMA 2.9. *For a generic ϕ_t, $\mathcal{M}(R_i, a)$ is a smooth manifold and*

$$\dim \mathcal{M}(R_i, a) = i - \mu(a) + \dim R_i.$$

LEMMA 2.10. *For a generic ϕ_t, the map $\Pi^{\ell}_{i,a}: \mathcal{M}(R_i, a) \to R_i$, defined by taking boundary value at $-\infty$, is smooth.*

Furthermore, for each i, j and a, the two maps $\Pi^{r}_{i,j}$, $\Pi^{\ell}_{j,a}$ are transversal to each other.

LEMMA 2.11. *For generic ϕ_t, we can find a compactification $\mathscr{C}\mathcal{M}_{\phi_t}(R_i, a)$ of $\mathcal{M}_{\phi_t}(R_i, a)$, which has a structure of compact smooth manifold with corners and which has the following properties:*

$$\mathrm{Int}\,\mathscr{C}\mathcal{M}_{\phi_t}(R_i, a) = \mathcal{M}_{\phi_t}(R_i, a), \tag{2.11.1}$$

$$\partial \mathscr{C}\mathcal{M}_{\phi_t}(R_j, a) = \bigcup_{j > i > k} \overline{\mathcal{M}_{\phi}(R_j, R_i) \times_{R_i} \mathcal{M}_{\phi_t}(R_i, a)} \cup \bigcup_{b} \overline{\mathcal{M}_{\phi_t}(R_j, b) \times_{R_i} \mathcal{M}_{\phi'}(b, a)}. \tag{2.11.2}$$

Now we use these moduli spaces to construct a chain homomorphism $\varphi: \hat{C}_*(M) \to C_*(M)$. We consider a subcomplex $C^{tg}_i(R_j)'$ of $C^{tg}_i(R_j)$ consisting of elements (P, f) transversal to $\Pi^{r}_{j,a}$. By Lemma 1.16, the inclusion $C^{tg}_i(R_j)' \to C^{tg}_i(R_j)$ induces an isomorphism on homology. Hence, by abuse of notation, we write $C^{tg}_i(R_j)$ in place of $C^{tg}_i(R_j)'$, hereafter. For $(P, f) \in C^{tg}_i(R_j)$, $a \in R_{\phi'}$ with $\mu(a) = i + j = k$, we put

$$\langle \varphi_k(X, f), a\rangle = \# X \times_{R_i} \mathcal{M}_{\phi_t}(R_i, a).$$

Here $\#$ is an order counted with sign. We define

$$\varphi_k(X, f) = \sum_a \langle \varphi_k(X, f), a\rangle [a]$$

In a way similar to the last section, we can use Lemma 2.11 to show:

LEMMA 2.12.

$$\sum_{k=1}^{n} \varphi_k \partial_{n-k} + \partial\varphi_n = 0.$$

By Lemma 2.12, we obtain a chain map

$$\varphi = \oplus\, \varphi_k : \hat{C}_n(M) \to C_n(M).$$

LEMMA 2.13. *Up to chain homotopy, the map φ is independent of the choice of the homotopy ϕ_t.*

The proof of Lemma 2.13 is a combination of the ideas we have already discussed and of a similar lemma used in the proof of well-definedness of Floer homology (see [13, Lemma 13.18]). We omit the proof. Next we choose a homotopy ϕ_t' from ϕ' to ϕ, and define $\mathcal{M}(a, R_i)$ in a similar way. Then we can define a chain map

$$\varphi' = \oplus\, \varphi_k' : C_n(M) \to \hat{C}_n(M).$$

Up to chain homotopy, this map is independent of the choice of ϕ_t'.

Finally, we shall prove $\varphi'\varphi$ and $\varphi\varphi'$ are chain map homotopic to identity. For this purpose, we first take a smooth function $\chi: \boldsymbol{R} \to [0, 1]$ such that

$$\chi(t) = \begin{cases} 0 & \text{if } t < -1 \\ 1 & \text{if } t > 1. \end{cases}$$

Take a sufficiently large positive number T and put

$$\phi_t'' = (1 - \chi(t))\phi_{t+T} + \chi(t)\phi_{t-T}'.$$

Then using a differential equation

$$F_A + *F_A + \operatorname{grad}_{a_t} f_{\phi_t''} \wedge dt - \bar{*} \operatorname{grad}_{a_t} f_{\phi_t''} = 0$$

we define moduli space $\mathcal{M}(R_i, R_j)'$ for sufficiently large T. (The transversality conditions we need are satisfied automatically in that case.) We remark that we did not divide by the $\boldsymbol{R}$-action. Using them we define a chain map

$$\varphi'' : \hat{C}_n(M) \to \hat{C}_n(M).$$

Then we can prove that this map is chain homotopy equivalence in the case when $\phi_t'' \equiv \phi$. (The proof is similar to one of Lemma 2.13.) Furthermore, we can prove that

$$\varphi'' = \varphi'\varphi$$

for sufficiently large T. (The proof of this fact is similar to the proof of [13, Lemma 13.12] and is omitted.) Thus, we proved that $\varphi'\varphi$ is chain homotopic to identity. In a similar way we can prove that $\varphi\varphi'$ is chain homotopic to identity. The proof of Proposition 2.7 is now complete.

3. SPECTRAL SEQUENCE FOR CONNECTED SUM 1

In this section, we apply the idea of the last section and construct the spectral sequence in our main theorem.

Let M_1 and M_2 be homology 3-spheres and $R(M_i)$, $i = 1, 2$ be as in Section 2. In general, $R(M_i)$ are of positive dimension and hence in the definition of Floer homology one needs a perturbation. But in this section, we assume that they are discrete and

$$H^1(M_i, su(2)^a) = 0 \tag{3.1}$$

for each $a \in su(2)$. To deal with general case, we can just start from the perturbed Chern–Simons functional and discuss in the same way. So Assumption (3.1) is not essential. We assume it only to save notation.

Now, we consider the connected sum $M = M_1 \# M_2$. We remark

$$\bar{R}(M) \cong \bar{R}(M_1) \times \bar{R}(M_2) \times SO(3) \cup \bar{R}(M_1) \cup \bar{R}(M_2). \tag{3.2}$$

In particular, $\bar{R}(M)$ has positive dimension although we assumed that $\bar{R}(M_i)$ are discrete. But we can prove easily that Assumptions 2.1 is satisfied in this case. (The self-indexing property will be a consequence of the discussions later.) The E^1-term of this spectral sequence is

$$(C_*(M_1) \otimes C_*(M_2) \otimes C_*^{gt}(SO(3))) \oplus C_*(M_1) \oplus C_*(M_2). \tag{3.3}$$

But this is not enough to prove our main theorem since we assert that we can control not only the E^1-term but also the E^2-term. The purpose of this section and the next is to find a relation between the differential ∂_1 of this spectral sequence and the boundary operators of $C_*(M_i)$. (We determine higher differential also.) By definition, the differential ∂_1 is obtained by using a moduli space of ASD-connections on $M \times \boldsymbol{R}$. So if we can find a relation between moduli space of ASD-connections on $M \times \boldsymbol{R}$ and one on $M_i \times \boldsymbol{R}$, we will get an answer. One might be able to do it by using a sequence of metrics on M such that, in the limit, M splits into M_1 and M_2. But unfortunately, the analysis one needs to realize this idea is difficult and so far the author could not do it in that way. Instead, in this section and the next, we proceed as follows. First we construct spaces $\mathscr{M}([a, b], [a', b'])$ for $a, a' \in R(M_1)$, $b, b' \in R(M_2)$, together with "boundary maps", Π^{ℓ}, Π^{r}, using only moduli spaces on M_i (and not on M). Next we show that the spaces and boundary maps satisfy the conclusions of Theorems 2.3–2.5 and Lemmas 2.9–2.11. Here we remark that the spaces $\mathscr{M}([a, b], [a', b'])$ are not directly regarded as sets of orbits of a gradient vector field but they have the same properties. Then the argument of the last two sections enables us to find a spectral sequence. The differential of this spectral sequence is determined by the moduli spaces on $M_i \times \boldsymbol{R}$, which completes the proof of our main theorem (modulo analytic lemmas proven in Part II).

To define $\mathscr{M}([a, b], [a', b'])$ we need some notations. First we remark that we may and will assume that, for $a, a' \in R(M_1)$, $b, b' \in R(M_2)$, the moduli spaces $\mathscr{M}(a, a')$, $\mathscr{M}(b, b')$ satisfy the conclusion of Theorem 2.3; namely,

$$\mathscr{M}(a, a') = \mu(a) - \mu(a') - 1$$

$$\mathscr{M}(b, b') = \mu(b) - \mu(b') - 1.$$

We put

$$\tilde{R}_i(M_1, M_2) = \bigcup_{\mu(a)+\mu(b)=i} \{a\} \times \{b\} \times SO(3) \cup \bigcup_{\mu(a)=i-8k} \{a\} \times \{\theta_k\} \cup \bigcup_{\mu(b)=8k-i} \{\theta_k\} \times \{b\}.$$

Here θ_0 is the trivial connection and θ_k is a connection gauge equivalent to it and $\mu(\theta_k) = 8k$. $\boldsymbol{Z}$ acts on $R(M_i)$ as deck transformations of covering $R(M_i) \to \bar{R}(M_i)$. Then

Z acts on $\tilde{R}(M_1, M_2)$ by $k \cdot (a, b, u) = (k \cdot a, -k \cdot b, u)$, $k \cdot (a, \theta_\ell) = (k \cdot a, \theta_{\ell - k})$. Let $R_i(M_1, M_2)$ be the quotient space of this action and $\pi: \tilde{R}_i(M_1, M_2) \to R_i(M_1, M_2)$ be the projection. We write R_i in place of $R_i(M_1, M_2)$ so that no confusion can occur. (This notation is consistent to the one in Section 1 and 2 because of (3.2).) We set

$$\mathscr{G}^0_{\ell,\delta}(M_i \times \boldsymbol{R}) = \left\{ g \in \mathscr{G}(\mathscr{M}_i \times \boldsymbol{R}) \,\middle|\, \lim_{t \to \pm\infty} g(x, t) = 1 \right\}$$

$$\tilde{\mathscr{M}}_{\ell,\delta}(a, a') \cong \tilde{\mathscr{M}}_{\ell,\delta}(a, a')/\mathscr{G}^0_{\ell,\delta}(M_i \times \boldsymbol{R}).$$

Then

$$\mathscr{G}_{\ell,\delta}(a, a')/\mathscr{G}^0_{\ell,\delta}(M_i \times \boldsymbol{R}) \cong \boldsymbol{Z}_2^2$$

and $\tilde{\mathscr{M}}_{\ell,\delta}(a, a')$ is a double covering of $\mathscr{M}_{\ell,\delta}(a, a')$ since the action of the diagonal $(-1, -1)$ on $\tilde{\mathscr{M}}_{\ell,\delta}(a, a')$ is trivial. So we have a $\boldsymbol{Z}_2$-action on $\tilde{\mathscr{M}}_{\ell,\delta}(a, a')$.

Next we define holonomy maps

$$\begin{aligned} h&: \tilde{\mathscr{M}}_{\ell,\delta}(a, a') \to SU(2) \\ h&: \tilde{\mathscr{M}}_{\ell,\delta}(b, b') \to SU(2). \end{aligned} \tag{3.4}$$

Take base points $p_i \in M_i$. Let ℓ_i be the curve $t \mapsto (t, p_i)$, $t \in \boldsymbol{R}$. Then, since the bundle we are considering is trivial, there are holonomy maps

$$\begin{aligned} h&: \tilde{\mathscr{M}}_{\ell,\delta}(a, a') \to SU(2) \\ h&: \tilde{\mathscr{M}}_{\ell,\delta}(b, b') \to SU(2). \end{aligned}$$

It is easy to see that this map induces the map (3.4). Now we put

$$\begin{aligned} &\mathscr{M}((a, b), (a', b')) = \emptyset \\ &\mathscr{M}((a, b), (a', b)) = (\tilde{\mathscr{M}}_{\ell,\delta}(a, a') \times SU(2))/\boldsymbol{Z}_2 \\ &\mathscr{M}((a, b), (a, b')) = (\tilde{\mathscr{M}}_{\ell,\delta}(b, b') \times SU(2))/\boldsymbol{Z}_2 \end{aligned}$$

if $a \neq a'$, $b \neq b'$. Here the $\boldsymbol{Z}_2$-action is the diagonal action (it acts on $SU(2)$ by $g \mapsto -g$). We define

$$\begin{aligned} \Pi^\ell_{(a,b),(a',b')}&: \mathscr{M}((a, b), (a', b')) \to \{a\} \times \{b\} \times SO(3) \\ \Pi^r_{(a,b),(a',b')}&: \mathscr{M}((a, b), (a', b')) \to \{a'\} \times \{b'\} \times SO(3) \end{aligned}$$

by

$$\begin{aligned} \Pi^r_{(a,b),(a',b)}([A], g) &= [g] \\ \Pi^\ell_{(a,b),(a',b)}([A], g) &= [h(A)^{-1} g] \\ \Pi^r_{(a,b),(a,b')}([A], g) &= [g] \\ \Pi^\ell_{(a,b),(a,b')}([A], g) &= [g h(B)]. \end{aligned}$$

Note that the maps $\Pi^r_{(a,b),(a',b')}$, $\Pi^\ell_{(a,b),(a',b')}$ are well defined as maps to $SO(3)$ but not to $SU(2)$. We define

$$\mathscr{M}_1(i, j) = \bigcup_{\pi(a,b) \in \tilde{R}_i,\ \pi(a',b') \in \tilde{R}_j} \mathscr{M}((a, b), (a', b')).$$

The maps

$$\Pi^{\ell,1}_{i,j}:\mathcal{M}_1(i,j)\to R_i$$
$$\Pi^{r,1}_{i,j}:\mathcal{M}_1(i,j)\to R_j$$

are defined by patching $\Pi^{\ell}_{(a,b),(a',b')}$ or $\Pi^{r}_{(a,b),(a',b')}$.

Next we consider the case when some of a, b, a', b' are trivial connections. First we remark that for each k the group

$$I_{\theta_k}=\{g\in\hat{\mathcal{G}}(M)\,|\,g^*\theta_k=\theta_k\}$$

is isomorphic to $SU(2)$. For each $g\in I_{\theta_k}$ we choose $\tilde{g}_\pm\in\mathcal{G}(M\times\boldsymbol{R})$ such that $\tilde{g}_\pm=g$ if $t<-1$ and $\tilde{g}_\pm=\pm 1$ if $t>1$. We also take $A_{\theta_k,a}$, an element defined in a way similar to $A_{a,b}$ in Section 2. We put

$$\mathcal{G}_{\ell,\delta}(M\times\boldsymbol{R})(\theta_k,a)=\{h\in\mathcal{G}(M\times\boldsymbol{R})\,|\,\exists u\in\Omega^0_{\ell,\delta}\ \exists g\in I_k(M_i)h\exp(-u)=\tilde{g}_\pm\}$$
$$\hat{\mathcal{B}}_{\ell,\delta}(\theta_k,a)=\{A \text{ is an } su(2)\text{-valued 1-form}\,|\,A_{\theta,a}-A\in\Omega^1_{\ell,\delta}\}$$
$$\hat{\mathcal{M}}_{\ell,\delta}(\theta_k,a)=\{A\in\hat{\mathcal{B}}_{\ell,\delta}(\theta_k,a)\,|\,F_A+*F_A=0\}$$
$$\mathcal{M}_{\ell,\delta}(\theta_k,a)=\hat{\mathcal{M}}_{\ell,\delta}(\theta_k,a)/\mathcal{G}_{\ell,\delta}(M\times\boldsymbol{R})(\theta_k,a)$$
$$\tilde{\mathcal{M}}_{\ell,\delta}(\theta_k,a)=\hat{\mathcal{M}}_{\ell,\delta}(\theta_k,a)/\mathcal{G}^0_{\ell,\delta}(M\times\boldsymbol{R}).$$

Then

$$\mathcal{G}_{\ell,\delta}(M\times\boldsymbol{R})(\theta_k,a)/\mathcal{G}^0_{\ell,\delta}(M\times\boldsymbol{R})\cong SU(2)\times\boldsymbol{Z}_2.$$

This group acts on $\tilde{\mathcal{M}}_{\ell,\delta}(\theta_k,a)$. Since the action of $(-1,-1)\in SU(2)\times\boldsymbol{Z}_2$ is trivial it follows that $SU(2)$ acts on $\tilde{\mathcal{M}}_{\ell,\delta}(\theta_k,a)$. Then

$$\tilde{\mathcal{M}}_{\ell,\delta}(\theta_k,a)/SU(2)\cong\mathcal{M}_{\ell,\delta}(\theta_k,a).$$

There exists also a holonomy map

$$h:\tilde{\mathcal{M}}_{\ell,\delta}(\theta_k,a)\to SU(2).$$

We recall

$$\dim\mathcal{M}_{\ell,\delta}(\theta_k,a)=8k-\mu(a)-1-3.$$

If we define $\mathcal{M}_{\ell,\delta}(a,\theta_k)$ in a similar way, then

$$\dim\mathcal{M}_{\ell,\delta}(a,\theta_k)=\mu(a)-8k-1.$$

We put

$$\mathcal{M}([\theta_k,b],[a,b])=\tilde{\mathcal{M}}_{\ell,\delta}(\theta_k,a)/\boldsymbol{Z}_2$$
$$\mathcal{M}([a,b],[\theta_k,b])=\tilde{\mathcal{M}}_{\ell,\delta}(a,\theta_k)/\boldsymbol{Z}_2$$
$$\mathcal{M}([a,\theta_k],[a',\theta_k])=\mathcal{M}_{\ell,\delta}(a,a')$$
$$\mathcal{M}([a,\theta_k],[a,b])=\tilde{\mathcal{M}}_{\ell,\delta}(\theta_k,b)/\boldsymbol{Z}_2$$
$$\mathcal{M}([a,b],[a,\theta_k])=\tilde{\mathcal{M}}_{\ell,\delta}(b,\theta_k)/\boldsymbol{Z}_2$$
$$\mathcal{M}([\theta_k,b],[\theta_k,b'])=\mathcal{M}_{\ell,\delta}(b,b').$$

The Z_2-action is the diagonal one. We define the maps Π^{r} and Π^{ℓ} by

$$\Pi^{\ell}_{[\theta_k,b],[a,b]}([A]) = 1, \qquad \Pi^{\mathrm{r}}_{[\theta_k,b],[a,b]}([A]) = h(A)$$

$$\Pi^{\ell}_{[a,b],[\theta_k,b]}([A]) = h(A)^{-1}, \qquad \Pi^{\mathrm{r}}_{[a,b],[\theta_k,b]}([A]) = 1$$

$$\Pi^{\ell}_{[a,\theta_k],[a',\theta_k]}([A]) = 1, \qquad \Pi^{\mathrm{r}}_{[\theta_k,b],[\theta_k,b']}([A]) = 1$$

$$\Pi^{\ell}_{[a,\theta_k],[a,b]}([B]) = 1, \qquad \Pi^{\mathrm{r}}_{[a,\theta_k],[a,b]}([B]) = h(B)^{-1}$$

$$\Pi^{\ell}_{[a,b],[a,\theta_k]}([B]) = h(B), \qquad \Pi^{\mathrm{r}}_{[a,b],[a,\theta_k]}([B]) = 1$$

$$\Pi^{\ell}_{[\theta_k,b],[\theta_k,b']}([B]) = 1, \qquad \Pi^{\mathrm{r}}_{[\theta_k,b],[\theta_k,b']}([B]) = 1.$$

By collecting these maps, we obtain $\Pi^{\ell}_{i,j}$ and $\Pi^{\mathrm{r}}_{i,j}$.

Now we are going to show that these spaces and maps have the properties we used in Section 2. For this purpose, we need to state Theorem 2.5 a bit more precisely. We put $I_a = \{\pm 1\}$ if a is irreducible. Then $I_a \times I_b$ acts on $\tilde{\mathcal{M}}(a,b)$ with $\tilde{\mathcal{M}}(a,b)/I_a \times I_b = \mathcal{M}(a,b)$. Let $\mathcal{M}(a, c_1,\ldots, c_n, b)$ be the quotient of

$$\tilde{\mathcal{M}}(a, c_1) \times \cdots \times \tilde{\mathcal{M}}(c_n, b)$$

by the diagonal action of $I_a \times \prod I_{c_i} \times I_b$. We put

$$\mathscr{C}\mathcal{M}(a,b) = \bigcup_{c_1,\ldots,c_n,\, n=1,2,\ldots} \mathcal{M}(a, c_1,\ldots, c_n, b).$$

We also put

$$\mathcal{M}(a,b;C) = \{[A] \in \mathcal{M}(a,b) \mid |F_A|_{C^0} < C\}.$$

THEOREM 3.4. *$\mathscr{C}\mathcal{M}(a,b)$ has a structure of orbitfold with corners and boundaries. $\mathcal{M}(a,b;C)$ is precompact in $\mathscr{C}\mathcal{M}(a,b)$ for each C. Moreover,*

$$\partial\mathcal{M}(a,b) = \bigcup_{I_c=\{\pm 1\}} \mathcal{M}(a,c,b).$$

In case $\mu(a) - \mu(b) < 8$, we have $\mathcal{M}(a,b;C) = \mathcal{M}(a,b)$ for sufficiently large C. Hence, $\mathcal{M}(a,b)$ itself is compact in that case.

For the proof of Theorem 3.4 we refer to [13]. Now as in Section 1, we put

$$\mathscr{C}\mathcal{M}(j,k) = \mathcal{M}(j,k) \cup \bigcup_{j>i_1\cdots i_n>k} \mathcal{M}(j,i_1) \times_{R_{i_1}} \cdots \times_{R_{i_n}} \mathcal{M}(i_n,k).$$

It follows immediately from Theorem 3.4 that

LEMMA 3.5. *$\mathscr{C}\mathcal{M}(a,b)$ has a structure of orbifold with corners and boundaries such that*

$$\partial\mathscr{C}\mathcal{M}(j,k) = \bigcup_{\ell} \mathcal{M}(j,\ell) \times_{R_\ell} \mathcal{M}(\ell,k).$$

If $j - k < 8$ then $\mathscr{C}\mathcal{M}(j,k)$ is compact.

We next see that the maps $\Pi^{\ell}_{i,j}$ and $\Pi^{\mathrm{r}}_{i,j}$ are compatible with compactification. We remark that there exist induced maps

$$\mathcal{M}(j,i_1) \times_{R_{i_1}} \cdots \times_{R_{i_n}} \mathcal{M}(i_n,k) \to R_j$$

$$\mathcal{M}(j,i_1) \times_{R_{i_1}} \cdots \times_{R_{i_n}} \mathcal{M}(i_n,k) \to R_k$$

which we write $\Pi^{\ell}_{j,i_1,\ldots,i_n,k}$, $\Pi^{\mathrm{r}}_{j,i_1,\ldots,i_n,k}$. Now we assert:

LEMMA 3.6. *The maps $\Pi^{\ell}_{j,i_1,\dots,i_n,k}$ are patched together to give a smooth map from $\mathscr{CM}(j,k)$ to R_j. The maps $\Pi^{r}_{j,i_1,\dots,i_n,k}$ are patched together to give a smooth map from $\mathscr{CM}(j,k)$ to R_k.*

Proof. Let $[A_m, g_m] \in \mathscr{M}((a,b),(a',b)) \subset \mathscr{M}(j,k)$ be a sequence converging to $[[A_{\infty,1}, g_{\infty,1}], [A_{\infty,2}, g_{\infty,2}]] \in \mathscr{M}(j,i,k)$. We will show

$$\lim_{m\to\infty} \Pi^{\ell}_{j,k}([A_m, g_m]) = \Pi^{\ell}_{i,k}([A_{\infty,1}, g_{\infty,1}]). \tag{3.7}$$

By definition, we have

$$\lim_{m\to\infty} h(A_m) = h(A_{\infty,2})h(A_{\infty,1}). \tag{3.8}$$

We first consider the case when the boundary values of A_i, etc., are all irreducible. Since $[[A_{\infty,1}, g_{\infty,1}], [A_{\infty,2}, g_{\infty,2}]] \in \mathscr{M}(j,i,k)$, we have

$$g_{\infty,1} = h(A_{\infty,2})^{-1} g_{\infty,2}.$$

Therefore, by (3.8),

$$\begin{aligned}
\Pi^{\ell}_{i,k}([A_{\infty,1}, g]) &= h(A_{\infty,1})^{-1} g_{\infty,1} \\
&= h(A_{\infty,1})^{-1} h(A_{\infty,2})^{-1} g_{\infty,2} \\
&= \lim_{m\to\infty} h(A_m)^{-1} g_m \\
&= \lim_{m\to\infty} \Pi^{r}_{i,k}([A_m, g_m, t_m]).
\end{aligned}$$

Thus, we have proved (3.7). The above argument can be summarized as in Fig. 1.

In a similar way, we can prove that the map is continuous between other strata (see Fig. 2).

The proof of smoothness is based on the decay estimate. To give a proof one has to go back to the proof of Lemma 3.5 (but once we recall it the proof is routine). We omit it.

Finally, we remark that, by using a perturbed Chern–Simons functional if necessary, one may assume that transversality between map $\Pi^{r}_{j,i}$ and $\Pi^{\ell}_{i,k}$ and similar transversality hold.

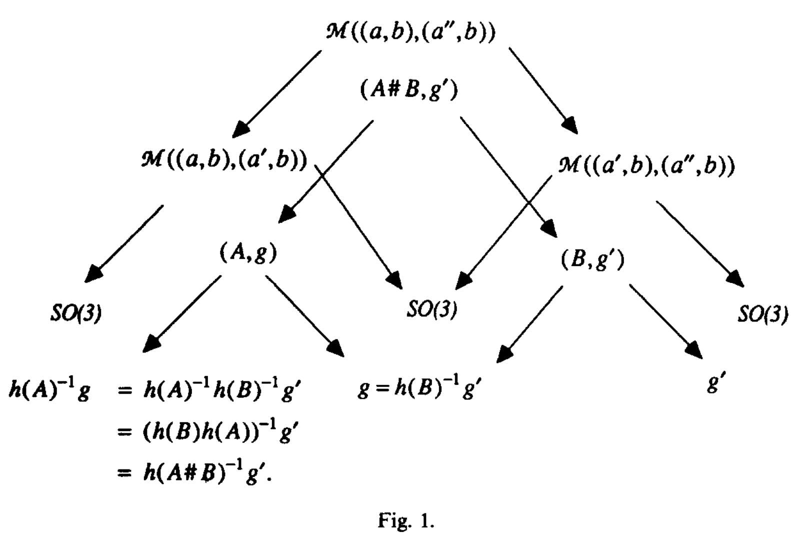


Fig. 1.

Thus, we have established all the properties we used in the last section. Hence, we have a chain complex $(\hat{C}, \hat{\partial})$ and a spectral sequence converging to the homology of $(\hat{C}, \hat{\partial})$. (In fact we need to discuss the orientation to check the sign. But since the argument is the same as the one in Section 1, we omit it.)

Finally, we calculate the E^2-term of the spectral sequence. By the same argument as in Section 2, we have

$$E^1_{i,j} = \bigoplus_{\substack{a\in R(M_1)\\ b\in R(M_2)\\ \mu(a)+\mu(b)=i,}} H_j(SO(3);\boldsymbol{Z})\cdot[a,b] \oplus \bigoplus_{\substack{a\in R(M_1)\\ \mu(a)=i}} H_j(*;\boldsymbol{Z})\cdot[a,\theta_0] \oplus \bigoplus_{\substack{b\in R(M_2)\\ \mu(b)=i}} H_j(*;\boldsymbol{Z})\cdot[\theta_0,b].$$

If $a, a' \in R(M_1)$, $b \in R(M_2)$ are irreducible with $\mu(a') = \mu(a) - 1$, then we have a diagram (see Fig. 3) where $\mathcal{M}((a,a'),b) \cong \bar{\mathcal{M}}(a,a') \times SO(3)$, and Π^{ℓ}, Π^{r} are identity on each connected component.

Therefore, the $H_j(SO(3);\boldsymbol{Z})\cdot[a,b] \to H_j(SO(3);\boldsymbol{Z})\cdot[a',b]$-component of ∂_1 is given by $u\cdot[a,b] \mapsto \langle\partial a, a'\rangle u\cdot[a',b]$. Similarly, the $H_j(SO(3);\boldsymbol{Z})\cdot[a,b] \to H_j(SO(3);\boldsymbol{Z})\cdot[a,b']$ component is given by $u\cdot[a,b] \mapsto \langle\partial b, b'\rangle u\cdot[a,b']$.

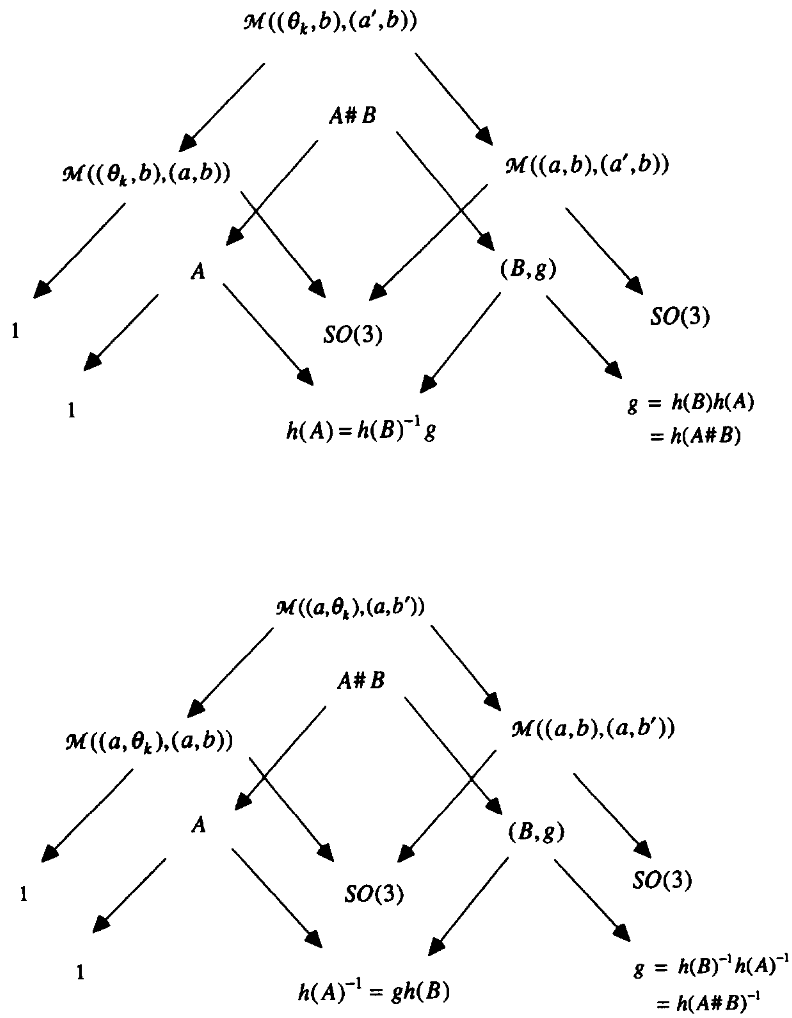
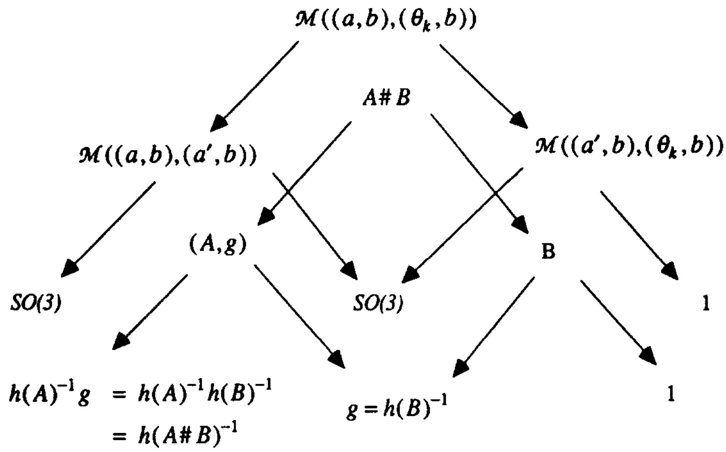


Fig. 2.



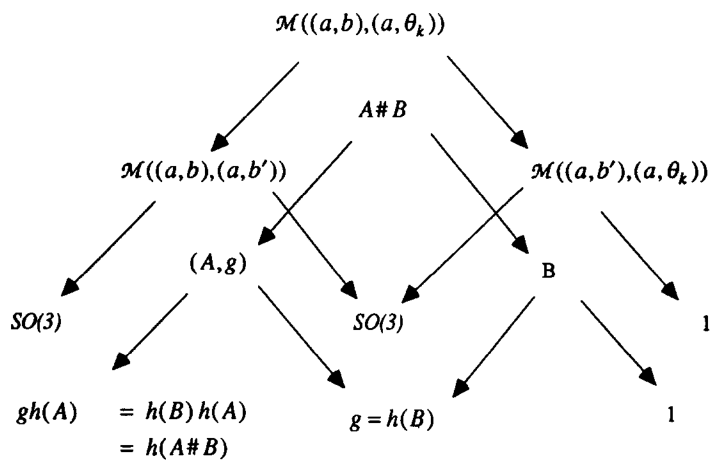


Fig. 2. (continued)

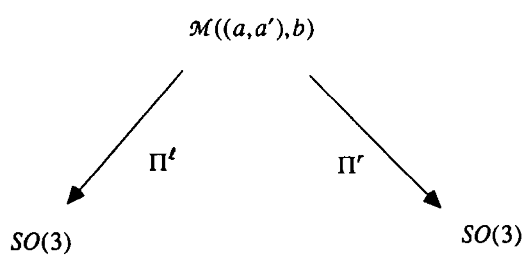


Fig. 3.

Next, if $a \in R(M_1)$, $b \in R(M_2)$ are irreducible connections with $\mu(a) = 1$, then the $H_0(*; \boldsymbol{Z}) \cdot [a, b] \to H_0(SO(3); \boldsymbol{Z}) \cdot [\theta_0, b]$ component of ∂_1 is given by $u \cdot [a, b] \mapsto \# \hat{\mathcal{M}}(a, \theta_0) u \cdot [\theta_0, b]$, where $\# \hat{\mathcal{M}}(a, \theta_0)$ is the order counted with sign. The $H_0(*; \boldsymbol{Z}) \cdot [a, \theta_0] \to H_0(SO(3); \boldsymbol{Z}) \cdot [a, b]$ component is given in a similar way.

Finally, the $H_0(*; \boldsymbol{Z}) \cdot [a, \theta_0] \to H_0(*; \boldsymbol{Z}) \cdot [a', \theta_0]$ and the $H_0(*; \boldsymbol{Z}) \cdot [\theta_0, b] \to H_0(*; \boldsymbol{Z}) \cdot [\theta_0, b']$ components of ∂_1 are multiplications, by $\langle \partial a, a' \rangle$ and $\langle \partial b, b' \rangle$, respectively. The other components of ∂_1 are 0.

Thus, we have an exact sequence of chain complex

$$0 \to (C_*(M_1), \partial) \oplus (C_*(M_2), \partial) \to (E^1_{*,0}, d^1) \to (C_*(M_1), \partial) \otimes (C_*(M_2), \partial) \to 0.$$

It induces a long exact sequence

$$\to I_i(M_1) \oplus I_i(M_2) \to E^2_{i,0} \to H_i((C_*(M_1), \partial) \otimes (C_*(M_2), \partial)) \to$$
$$\to I_{i-1}(M_1) \oplus I_{i-1}(M_2) \to E^2_{i-1,0} \to H_{i-1}((C_*(M_1), \partial) \otimes (C_*(M_2), \partial)) \to$$

For $j \neq 0$, we have

$$E^2_{ij} \cong H_i((C_*(M_1), \partial) \otimes (C_*(M_2), \partial); H_j(SO(3); \boldsymbol{Z})).$$

Thus, we have constructed the spectral sequence in our main theorem. We shall prove in next section that the homology of $(\hat{C}, \hat{\partial})$ is isomorphic to the Floer homology of $M = M_1 \# M_2$.

4. SPECTRAL SEQUENCE FOR CONNECTED SUM II

In this section, we show that the chain complex in Section 3 gives the Floer homology of the connected sum of M_1 and M_2.

As in Section 3, the basic idea is the combination of the argument of Section 2 and some analytic lemmas.

Let M_1 and M_2, etc., be as in Section 3, and M be the connected sum of M_1 and M_2. As we remarked there, the representation space $\bar{R}(M)$ is of positive dimension. Hence, according to the definition of Floer homology, we take a perturbation of Chern–Simons functional. In order to save notation, we denote by $\bar{R}(M)$ the set of critical values of this perturbed functional, and put

$$R(M) = \bar{R}(M) \times \boldsymbol{Z}.$$

For each element α of $R(M)$, its Morse index $\mu(\alpha)$ is defined such that the moduli space $\mathcal{M}(\alpha, \beta)$ of perturbed ASD equation satisfies

$$\dim \mathcal{M}(\alpha, \beta) = \mu(\alpha) - \mu(\beta) - 1 - \dim I_\alpha.$$

In order to apply the argument of Section 2, we need to define spaces, $\mathcal{M}(R_i, \alpha)$, where R_i is as in Section 3, and also we need to define a map

$$\Pi^\ell_{i,\alpha} : \mathcal{M}(R_i, \alpha), \to R_i.$$

We recall that the connected components of R_i are either one point or $SO(3)$. Hence, we get

$$\Pi^\ell_{(a,b),\alpha} : \mathcal{M}((a, b), \alpha) \to SO(3)$$

for $a \in R(M_1)$, $b \in R(M_2)$.

Let N' be the boundary connected sum of $M_1 \times [0, 1]$ and $M_2 \times [0, 1]$ along $M_1 \times \{1\}$, $M_2 \times \{1\}$. We define a metric on $N = \operatorname{Int} N'$ such that N minus a compact set is isometric to the disjoint union of $M_1 \times (-\infty, 0)$, $M_2 \times (-\infty, 0)$, and $M \times (0, \infty)$. Define Sobolev spaces $\Omega^0_{\ell,\delta}(N)$, $\Omega^1_{\ell,\delta}(N)$, as in Section 2.

As in Section 3, we assume that $R(M_i)$ is discrete. Fix a lift $s: R(M_i) \to \mathcal{A}(M_i)$. For each $a \in U_1$, $b \in U_2$, $[\alpha] \in R(M)$, there exists a connection $A_{a,b,\alpha}$ on N, which coincides with $s(a)$, $s(b)$, α, respectively, outside a compact set. We put

$$\hat{\mathcal{B}}_{\ell,\delta}((a, b), \alpha) = \{A = \Omega^1(N) \mid A_{a,b,\alpha} - A \in \Omega^1_{\ell,\delta}\}.$$

Define a Gauge group $\mathscr{G}_{\ell+1,\delta}((a,b),\alpha)$ as in Section 2. It acts on $\hat{\mathscr{B}}_{\ell,\delta}((a,b),\alpha)$. Let $\mathscr{B}_{\ell,\delta}((a,b),\alpha)$ be the quotient space of this action. We put

$$\mathscr{M}_{\ell,\delta}((a,b),\alpha) = \{[A] \in \mathscr{B}_{\ell,\delta}((a,b),\alpha) \mid *F_A = -F_A\}.$$

Since $\mathscr{M}_{\ell,\delta}((a,b),\alpha)$ is independent of ℓ, δ, we omit these suffixes so that no confusion can occur. As in Section 2, our perturbation was invariant by the $\boldsymbol{R}$-action induced by translation, hence was not of compact support. Hence, we cannot take perturbation of compact support. Again we defer to Part II the precise argument, and just mention that there is a family by perturbations of the ASD equations parametrized by $\psi' \in \Psi'$. The perturbation is of compact support. Let $\mathscr{M}_{\psi'}((a,b),\alpha)$ be the moduli space of the perturbed ASD equation. Now we have the following.

THEOREM 4.1. *$\mathscr{M}_{\psi'}((a,b),\alpha)$ is a smooth manifold and*

$$\dim \mathscr{M}_{\psi'}((a,b),\alpha) = \mu(a) + \mu(b) - \mu(\alpha) - \dim I_a - \dim I_b + 3$$

for each element ψ' of a residual subset of Ψ'.

The proof is deferred to Part II. Hereafter we omit ψ'.

Now we define the map

$$\Pi^{\ell}_{(a,b),\alpha}: \mathscr{M}((a,b),\alpha) \to SO(3).$$

We use holonomy representation for this purpose. Let $\ell:(-\infty,\infty) \to N$ be a curve such that

$$\ell(t) = \begin{cases} (t+T, p_2) & \text{if } t < -T \\ (t-T, p_2) & \text{if } t > T. \end{cases}$$

Here $p_i \in M_i$, as in the definition of the map h in Section 3 (note $M_i \times (-\infty, 0) \subset N$). In case $a \in R(M_1)$, $b \in R(M_2)$ are both irreducible, one obtains a map

$$h: \mathscr{M}((a,b),\alpha) \to SO(3)$$

by taking a holonomy along ℓ. We define

$$\Pi^{\ell}_{(a,b),\alpha}([A], g) = h(A).$$

In order to prove that $\mathscr{M}((a,b),\alpha)$, $\Pi^{\ell}_{(a,b),\alpha}$ have the properties we used in Section 2, we need the following lemma. We put

$$\begin{aligned}\mathscr{C}\mathscr{M}((a,b),\alpha) = \mathscr{M}((a,b),\alpha) \cup \bigcup_{c_i \in R(M_1)} \bigcup_{c'_i \in R(M_2)} \bigcup_{c''_i \in R(M_3)} &\{\mathscr{M}(a,c_1) \times \cdots \times \mathscr{M}(c_{k-1},c_k) \\ &\times \mathscr{M}(b,c'_1) \times \cdots \times \mathscr{M}(c'_{k'-1},c'_{k'})\} \\ &\times \mathscr{M}((c_k,c'_{k'}),c''_1) \times \cdots \times \mathscr{M}(c''_{k''},b).\end{aligned}$$

The maps Π^{ℓ} on $\mathscr{C}\mathscr{M}((a,b),\alpha)$ are defined by

$$\Pi^{\ell}((A_1,\ldots,A_k),(A'_1,\ldots,A'_k),(A''_1,\ldots,A''_k)) = h(A_1)^{-1}\cdots h(A_k)^{-1}h(A''_1)h(A'_{k'})\cdots h(A'_1).$$

LEMMA 4.2. *$\mathscr{C}\mathscr{M}((a,b),\alpha)$ has a structure of smooth orbifold with corners. Π^{ℓ} is a smooth map on it.*

The proof is similar to Lemma 3.6 and is omitted.

Using these moduli spaces we can construct a chain map φ from $(\hat{C}, \hat{\partial})$ to (C, ∂). Here $(\hat{C}, \hat{\partial})$ is the chain complex as in the last section and (C, ∂) is the chain complex used to define the Floer homology $I_*(M)$.

Let $-N$ be the manifold N with the opposite orientation. Using the moduli spaces $\mathcal{M}(\alpha, (a, b))$ of ASD connections on $-N$ in place of $\mathcal{M}((a, b), \alpha)$ we can define a chain map ψ from (C, ∂) to $(\hat{C}, \hat{\partial})$. Now we show that the maps on homology induced by the two maps φ and ψ are inverse to each other.

First we consider $\psi\varphi$. For this purpose, we consider the family of 4-manifold $N(T)$ obtained by patching $(N - M \times (T, \infty)) \cup ((-N) - M \times (-\infty, -T))$ along $M = M \times \{T\} = M \times \{-T\}$. For $a, a' \in R(M_1)$, $b, b' \in R(M_2)$, we let $\mathcal{M}((a, a'), (b, b'))$ denote the moduli space of all ASD connections on $N(T)$ whose boundary values are a, a', b, b' at $M_1 \times \{-\infty\}$, $M_1 \times \{+\infty\}$, $M_2 \times \{-\infty\}$, $M_2 \times \{+\infty\}$, respectively.

We remark that $N(T)$ is the connected sum of $M_1 \times \boldsymbol{R}$ and $M_2 \times \boldsymbol{R}$. Hence, Taubes' argument, which is now standard in Gauge theory, makes it possible to find a cobordism between $\mathcal{N}((a, b), (a', b'))$ and $(\mathcal{M}(a, b) \times \mathcal{M}(a', b')) \tilde{\times} SO(3)$. Let $h_\pm : \mathcal{N}((a, b), (a', b')) \to SO(3)$ denote the holonomy map along the curve $\ell_\pm$ in Fig. 4.

We define

$$\Pi^{\ell}_{(a,b),(a',b')} : \mathcal{N}((a, b), (a', b')) \to SO(3)$$

$$\Pi^{r}_{(a,b),(a',b')} : \mathcal{N}((a, b), (a', b')) \to SO(3)$$

by using h_-, h_+ respectively. By restricting them, we obtain maps from $\mathcal{M}((a, b), (a', b'))$ to $SO(3)$, which we denote by the same symbol. Using $\mathcal{M}((a, b), (a', b'))$ and $\Pi^{\ell}_{((a,b),(a',b'))}$, $\Pi^{r}_{((a,b),(a',b'))}$, we obtain a chain map $\phi : (\hat{C}, \hat{\partial}) \to (\hat{C}, \hat{\partial})$. It is also standard to show that the composition $\psi\varphi$ is chain homotopic to the map ϕ.

We next show that ϕ is chain homotopic to the identity. For this purpose, we remark that if $\mu(a) + \mu(b) = \mu(a') + \mu(b')$ then $\mathcal{M}(a, b) \times \mathcal{M}(a', b')$ is nonempty only if $a = a'$, $b = b'$. In that case, $(\mathcal{M}(a, b) \times \mathcal{M}(a', b')) \tilde{\times} SO(3)$ is diffeomorphic to $SO(3)$. Furthermore, the restriction of $\Pi^{\ell}_{(a,b),(a',b')}$, $\Pi^{r}_{(a,b),(a',b')}$ coincides with $g \mapsto g^{-1}$ or $g \mapsto g$. (To see this we remark that the parameter $SO(3)$ represents the patching map on the neck of the connected sum. Then by the definition of the curves ℓ_+, ℓ_- the maps $\Pi^{\ell}_{(a,b),(a',b')}$, $\Pi^{r}_{(a,b),(a',b')}$ are as above.) Therefore, the chain map induced by $\mathcal{M}((a, b), (a', b'))$ and $\Pi^{\ell}_{(a,b),(a',b')}$, $\Pi^{r}_{(a,b),(a',b')}$ is

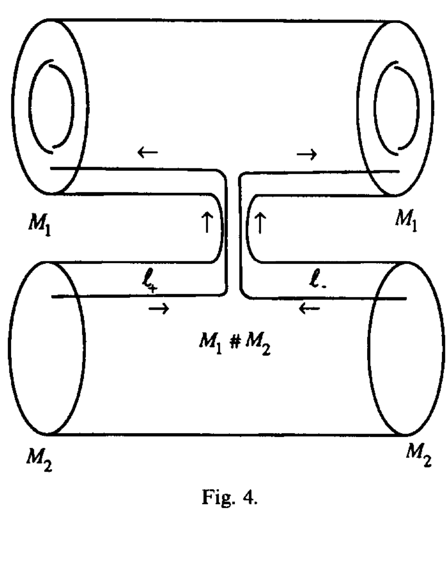


Fig. 4.

the identity map. On the other hand, by an argument similar to, for example, [13], the cobordism $\mathcal{N}((a,b),(a',b'))$ and the maps $\Pi^{\ell}_{(a,b),(a',b')}$, $\Pi^{r}_{(a,b),(a',b')}$ on it induce a chain homotopy between ϕ and the identity.

Thus, we have proved that $\psi\varphi$ is chain homotopic to identity.

We turn to prove that $\varphi\psi$ is chain homotopic to the identity. For this purpose, we use the 4-manifold $N'(T)$ obtained by patching $N-(M_1 \cup M_2)\times(-\infty,-T)$ and $(-N)-(M_1 \cup M_2)\times(T,\infty)$ along $M_1 \cup M_2$. We remark that the surgery of $N'(T)$ along the loop $\ell_+ \cup \ell_-$ is diffeomorphic to $(M_1 \# M_2)\times \mathbf{R}$. Hence, we need to study the relation of 1-dimensional surgery and the moduli spaces of ASD connections (see Fig. 5). We will prove Lemma 4.5 in Part II.

Let X be an oriented 4-manifold and ℓ be a loop on it. Suppose that there exists a compact subset K of X and a 3-manifold Y (not necessarily connected) such that $X-K$ is isometric to $Y\times(0,\infty)$. Let $R(Y)$ be the set of all flat connections on Y. We assume (for simplicity) that $R(Y)$ is discrete and satisfies (3.1). We take an $SU(2)$ bundle E on X equipped with a trivialization on M and ℓ. Let X' be the 4-manifold obtained from X by the surgery along ℓ. For $\alpha\in R(Y)$, let $\hat{\mathcal{M}}(X,\alpha)$ be the set of all ASD connections on X with boundary value α at infinity. We divide it by the group of gauge transformations which converges to identity at infinity and which is identity at a fixed base point p_0 on ℓ. There is a free $SU(2)$ action on it induced by the gauge transformation at p_0. Let $\tilde{\mathcal{M}}(X,\alpha)$ be the quotient space. We divide $\hat{\mathcal{M}}(X,\alpha)$ in the same way and let $\mathcal{M}(X',\alpha)$ be its quotient by the action of the group of gauge transformations which is identity at infinity. By taking a holonomy along ℓ we obtain a map $h:\tilde{\mathcal{M}}(X,\alpha)\to SO(3)$. We assume that transversality conditions are satisfied for $\tilde{\mathcal{M}}(X,\alpha)$ and for $\mathcal{M}(X',\alpha)$. We also assume that $1\in SO(3)$ is a regular value of h. Now we assert the following.

LEMMA 4.5. *There exists an oriented cobordism $\tilde{\mathcal{N}}(X,\alpha)$ between $\tilde{\mathcal{M}}(X',\alpha)$ and $h^{-1}(1)\subset\tilde{\mathcal{M}}(X,\alpha)$.*

The proof is deferred to Part II. Now we are ready to show that $\varphi\psi$ is chain homotopic to identity. For $\mu(\alpha)=\mu(\beta)$, we consider the fiber product

$$\bigcup_{(a,b)} \mathcal{M}(\alpha,(a,b))\times_{SO(3)}\mathcal{M}((a,b),\beta). \tag{4.6}$$

Here the maps $\mathcal{M}(\alpha,(a,b))\to SO(3)$ and $\mathcal{M}((a,b),\beta)\to SO(3)$ are Π^{r} and Π^{ℓ}, respectively. We let $\langle\phi'(\alpha),\beta\rangle$ be the order of the set (4.6) counted with sign. By definition, $\varphi\psi$ is chain homotopic to ϕ'.

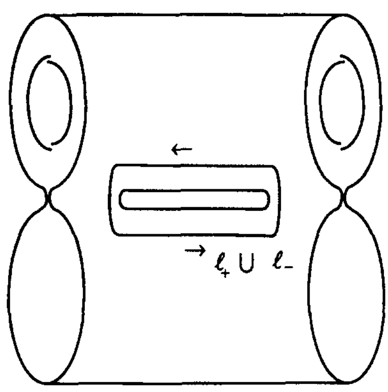


Fig. 5.

Now we put $X = N'(T)$ and $\ell = \ell_+ \cup \ell_-$ in Lemma 4.5. By definition and a standard patching argument the set (4.6) is cobordant to $h^{-1}(1)$. (In fact, we need to use the moduli space of perturbed ASD connections. But Lemma 4.5 holds without change in that case.) On the other hand, the surgery of $X = N'(T)$ along ℓ is $(M_1 \# M_2) \times \boldsymbol{R}$. Thus by Lemma 4.4, we conclude that $\varphi\psi$ is chain homotopic to identity.

We thus proved that the homology of the chain complex $(\hat{C}, \hat{\partial})$ is isomorphic to the one of (C, ∂). The proof of Theorem 0.0 modulo analytic lemmas is now complete.

5. AN EXAMPLE

Let M be the Poincaré homology sphere. It is well known that its fundamental group $\pi_1 M$ has exactly two nontrivial representations to $SU(2)$. It is proved in [10] that their degrees in Floer homology are 0 and 4, respectively (see Section 3 for our sign convention). Regarding them as elements in $R(M)$ they have degree $\equiv 0$ or $4 \bmod 8$. The trivial connections, as elements in $R(M)$, have degree $\equiv 0 \bmod 8$. We write ρ_k the element of degree k. Here $k \equiv 0$ or $4 \bmod 8$, and let θ_0 be the trivial connection. Similarly, $I_*(-M)$ has two generators in degree 1 and 5. Let τ_k ($k = 1, 5$) be the flat representations in degree k.

We consider the moduli spaces $\mathcal{M}(\rho_0, \rho_4)$ and $\mathcal{M}(\rho_4, \rho_0)$. Both are of dimension 3. The structure of these moduli spaces are determined by using ADHM constructions in [17]. In order to determine differentials in our spectral sequence we need to study the holonomy maps $h: \mathcal{M}(\rho_4, \rho_0) \to SO(3)$ and $h': \mathcal{M}(\rho_0, \rho_4) \to SO(3)$. We use the following unpublished result by P. Kronheimer.

Fact 5.1 (Kronheimer). The degrees of the maps h, h' are both nonzero.

In this section we determine the Floer homologies over $\boldsymbol{Q}$ of $M \# M$, and of $M \# -M$, by using Fact 5.1.

We first calculate the maps

$$H_i((C_*(M), \partial) \otimes (C_*(M), \partial)) \to I_{i-1}(M) \oplus I_{i-1}(M)$$
$$H_i((C_*(M), \partial) \otimes (C_*(-M), \partial)) \to I_{i-1}(M) \oplus I_{i-1}(-M) \tag{5.2}$$

in exact sequence (0.0.3). First we consider the situation of Theorem 0.0 in general and recall the following.

FOLK THEOREM 5.3 (see [9, 14]). *Let M be as in Theorem* 0.0. *Suppose that there exists a smooth* 4-*manifold X which bounds $-M$ such that the intersection form on $H^2(X; \boldsymbol{Z})$ is negative definite but is nondiagonalizable. Then the boundary maps* (5.2) *are nonzero.*

Proof (sketch). Suppose $\partial X = -M$. More precisely, we assume that X minus a compact set is diffeomorphic to $M \times (-\infty, 0)$. We consider the map

$$q: I_1(M) \to \boldsymbol{Z}, \qquad q(a) = \#\mathcal{M}(a, \theta_0)$$

we discussed in Section 0. By applying the argument of Donaldson [6] and Taubes [23] we find that the moduli spaces of ASD connections of instanton number 1 on X with boundary condition θ_0 is a space of dimension 5 such that its end is diffeomorphic to

$$\left(\overline{X \cup \bigcup_{\mu(a)=1} \tilde{\mathcal{M}}(X, a) \times \mathcal{M}(a; \theta_0)}\right) \times (0, \infty)$$

as an oriented manifold. Furthermore, each singularity of it is diffeomorphic to the cone of complex projective space and the number of the singular point is equal to the number of elements u of $H^2(X;\mathbf{Z})$ such that $(u\cdot u)\cap[X]=1$. Since the intersection form of $H^2(X;\mathbf{Z})$ is nontrivial, this number cannot be equal to the second Betti number of X. It follows that the signature of the manifold

$$\overline{X \cup \bigcup_{\mu(a)=1} \tilde{\mathcal{M}}(X,a)\times\mathcal{M}(a;\theta_0)}$$

is not equal to the second Betti number of X. Hence, the map $q\colon I_1(M)\to\mathbf{Z}$ is nonzero. It follows from the definition that the maps (5.2) are nonzero.

Thus, we find the E^2-term in case $M\#-M$. For $M\#M$ the boundary operator (0.1) is zero by degree counting.

We next turn to the calculation of the fourth derivative (which is the only nontrivial one since we work over rational coefficient).

In case of $M\#M$ we need to calculate the two maps

$$I_i(M)\oplus I_j(M)\to I_{i-4}(M)\otimes I_j(M)\otimes e_3\oplus I_i(M)\otimes I_{j-4}(M)\otimes e_3$$

$$I_i(M)\otimes I_j(M)\to I_{i-4}(M)\otimes I_j(M)\otimes e_3\oplus I_i(M)\otimes I_{j-4}(M)\otimes e_3.$$

Here e_3 is the generator of $H_3(SO(3);\mathbf{Q})$. The first one is obtained by

$$(x,y)\to O_{-4}(M)\otimes y\otimes e_3 + x\otimes O_{-4}(M)\otimes e_3.$$

Here

$$O_{-4}(M)=\sum_{\mu(a)=-4}\#\mathcal{M}(\theta_0,a)\cdot[a].$$

By the same argument as in Folk Theorem 5.3 we can prove that this element is nonzero. Therefore, we have

$$\partial_4(\rho_i,\theta_0)=c\rho_i\otimes\rho_{-4}$$

$$\partial_4(\theta_0,\rho_i)=c\rho_{-4}\otimes\rho_i$$

with $c\neq 0$. Next by Fact 5.1 and the discussion of the preceding two sections we have

$$\partial_4(\rho_i\otimes\rho_j\otimes e_0)=c_i\cdot\rho_{i-4}\otimes\rho_j\otimes e_3+c_j\cdot\rho_i\otimes\rho_{j-4}\otimes e_3$$

with $c_i, c_j\neq 0$. Here e_0 is a generator of $H_0(SO(3);\mathbf{Q})$.

Hence E^5 is generated by

$$\rho_4\otimes\theta_0-\theta_0\otimes\rho_4$$

$$\rho_4\otimes\rho_0\otimes e_0-\rho_0\otimes\rho_4\otimes e_0$$

$$2c\rho_0\otimes\rho_0\otimes e_0-c_0(\rho_0\otimes\theta_0+\theta_0\otimes\rho_0)$$

$$2c_4\rho_4\otimes\rho_4\otimes e_0-c_0(\rho_0\otimes\theta_0+\theta_0\otimes\rho_0).$$

There are no more differentials. Thus we have:

PROPOSITION 5.4.

$$I_*(M\#M;\mathbf{Q})\cong\begin{cases}\mathbf{Q}^2 & *\equiv 0,4 \bmod 8\\ 0 & \textit{otherwise.}\end{cases}$$

Next we turn to the case of $M\# -M$. By Folk Theorem 5.3, the generator of the E^2-term is

$$\begin{aligned} &\theta_0 \otimes \tau_j && (j = 1, 5) \\ &\rho_i \otimes \tau_1 \otimes e_3, && (i = 0, 4) \\ &\rho_i \otimes \tau_5 \otimes e_0, && (i = 0, 4) \\ &\rho_i \otimes \tau_5 \otimes e_3, && (i = 0, 4). \end{aligned}$$

The fourth differential is given by

$$\partial_4(\rho_i \otimes \tau_5 \otimes e_0) = c_i \rho_{i-4} \otimes \tau_5 \otimes e_3 + c_5 \rho_i \otimes \tau_1 \otimes e_3$$

$$\partial_4(\theta_0 \otimes \tau_j) = c\rho_4 \otimes \tau_j \otimes e_3.$$

Therefore we have:

PROPOSITION 5.5.

$$I_*(M\# -M; Q) \cong 0.$$

We remark that the fundamental group of $M\# M$ is isomorphic to that of $M\# -M$. But their Floer homologies are quite different to each other. By a similar calculation, one may calculate $I_*(kM\# -\ell M; Q)$ in general. For example,

$$\partial_1 = q \otimes id \otimes id \pm id \otimes q \otimes id \pm id \otimes id \otimes q$$

$$\partial_4(a \otimes b \otimes c \otimes e_0) = (\partial_4 a \otimes b \otimes c \pm a \otimes \partial_4 b \otimes c \pm a \otimes b \otimes \partial_4 c) \otimes e_3,$$

etc. The calculations are left for the reader.

PART II: ANALYSIS

6. PERTURBATION AND TRANSVERSALITY

In this section we study the perturbation of the ASD equation and prove Theorems 2.3, 2.4 and 4.1. First we consider the situation of Theorem 4.1 where we consider a 4-manifold with boundary and a perturbation of Riemannian metric. Let M be a 3-dimensional manifold satisfying Assumption 2.1. We choose and fix a Riemannian metric on it. Let N be a complete Reimannian 4-manifold such that N minus a compact set K is isometric to $M \times (0, \infty)$. We take a trivial $SU(2)$-bundle on N. For an open subset U on $R(M)$, we define $\mathcal{M}(N; U)$ as in Section 2 (the precise definition will be given later). This moduli space depends on the choice of metric g on K (note that the metric on the complement of K is fixed). We write $\mathcal{M}_g(N; U)$ in case we need to specify g explicitly. We put the C^∞-topology on the set of metrics on K. Let $\mathcal{M}e\ell(K)$ denote this set. We first prove the following:

THEOREM 6.1 (cf [25]). *If g is contained in a residual subset of $\mathcal{M}e\ell(K)$, then $\mathcal{M}_g(N, R(M))$ is a smooth manifold.*

Proof. First we fix notations. We write $\Omega^i_{\ell,\delta}$, $i = 0, 1, 2$, for the set of i-forms such that

$$\sum_{k \le \ell} \int_N e_\delta(x) |\nabla^k u|^p \, dx < \infty.$$

Here $e_\delta(x)$ is a smooth function on N such that

$$e_\delta(x)\begin{cases} = e^{\delta t} & \text{if } x=(v,t)\in M\times(1,\infty)\\ >0 & \text{everywhere.}\end{cases}$$

In case $i=2$, we require that u is self-dual in addition. Using $\Omega^i_{\ell,\delta}$, we define $\mathscr{G}_{\ell,\delta}(N)$, $\mathscr{B}_{\ell,\delta}(N;U)$ in a way similar to Section 2 (here U is an open subset of $R(M)$). For an element of $\mathscr{B}_{\ell,\delta}(N;U)$, we write (A,a), where A is a connection on N, and a is its boundary value contained in U. $\mathscr{M}(N;U)$ is the set of all gauge equivalence classes of ASD connections contained in $\mathscr{B}_{\ell,\delta}(N;U)$. In case $N=M\times\boldsymbol{R}$, U, $V\subseteq R(M)$ we define $\mathscr{B}_{\ell,\delta}(M\times\boldsymbol{R};U,V)$ in the same way (an element of it is expressed by $[A,a,b]$, etc.). We put

$$\mathscr{H}^1_{\ell,\delta}(N,a)=\left\{(B,b)\;\middle|\;\begin{array}{l} B \text{ is a 1-form on } N\\ b \text{ is a 1-form on } M\\ d_a^*b=0\\ B-A_b\in\Omega^1_{\ell,\delta}\end{array}\right\}.$$

Here A_b is a 1-form on N which coincides with b outside K. By an argument similar to [12], we can identify the tangent space of $\mathscr{B}_{\ell,\delta}(N;U)$ at (A,a) as

$$T_{[A,a]}(\mathscr{B}_{\ell,\delta}(N;U))=\{(B,b)\in\mathscr{H}^1_{\ell,\delta}(N,a)\,|\,d_A^*B=0\}.$$

In case the transversality condition is satisfied, the tangent space of $\mathscr{M}(N;U)$ at (A,a) is isomorphic to

$$\left\{(B,b)\in\mathscr{H}^1_{\ell,\delta}(N,a)\;\middle|\;\begin{array}{l} d_A^*B=0\\ *d_AB+d_AB=0\end{array}\right\}.$$

Hence, to prove Theorem 6.1, it suffices to show that the map

$$P_{A,a}:\mathscr{H}^1_{\ell,\delta}(N,a)\to\Omega^0_{\ell,\delta}\oplus\Omega^2_{\ell,\delta};\qquad (B,b)\mapsto(d_A^*B,\,*d_AB+d_AB)$$

is surjective for each (A,a) (if g is contained in a residual subset of $\mathscr{M}et(K)$). We take an (∞-dimensional) vector bundle $\mathscr{L}_{\ell,\delta}(N;U)$, $\mathscr{A}_{\ell,\delta}(N;U)\times_{\mathscr{G}_{\ell+1,\delta}(N)}\Omega^2_{\ell,\delta}\to\mathscr{B}_{\ell,\delta}(N;U)$ and consider a section to the direct product, $\mathscr{L}_{\ell,\delta}(N;U)\times\mathscr{M}et(K)$,

$$P^+:\mathscr{B}_{\ell,\delta}(N;U)\times\mathscr{M}et(K)\to\Omega^2_{\ell,\delta};\qquad ((A,a),g)\mapsto *_gF_A+F_A.$$

Here $*_g$ is the Hodge $*$-operator associated with the metric g. By the same argument as [12], the projection to the fiber of the differential of P^+ is surjective.

On the other hand, the differential of the restriction of P^+ to $\mathscr{B}_{\ell,\delta}(N;U)\times\{g\}$ is

$$(B,b)\mapsto *_gd_AB+d_AB.$$

Therefore, we can prove the surjectivity of $P_{A,a}$ by the same argument as [11].

Hence,

$$\{((A,a),g)\in\mathscr{B}(N,U)\times\mathscr{M}et\,|\,*_gF_A+F_A=0\}=\mathscr{M}od$$

is a smooth manifold. Since $P_{A,a}$ is a Fredholm operator for each (A,a) and g, it follows that the projection $\mathscr{M}od\to\mathscr{M}et$ is a Fredholm map. Hence as in [12], we can apply the Sard–Smale theorem to show that the set of regular values $g\in\mathscr{M}et$ is residual. For such g, $\{(A,a)\,|\,*_gF_A+F_A=0\}$ is smooth and $P_{A,a}$ is surjective for each (A,a) in this set. The proof of Theorem 6.1 is now complete.

Next we turn to the situation of Section 2. In this case we have to study the $\boldsymbol{R}$-invariant perturbation of the ASD equation on $M\times\boldsymbol{R}$. Hence, we cannot take a perturbation of metric with compact support. Instead, we perturb using holonomy representations, as in [7, 11] (see also [13]). We use the same symbol as in [13]. We recall the notations there, for reader's convenience. Let $p_0 \in M$ and $v_0 \in T_{p_0}M$. Choose an embedding $I: D^2 \to M$ such that $I(0) = p_0$. Let $\Gamma_1(p_0, I, v_0)$ be the set of smooth embeddings $\ell: S^1 \times D^2 \to M$ such that $\ell(0,0) = p_0$, $(D\ell/dt)(0,0) = v_0$, and $\ell(0,x) = I(x)$. Here we regard $S^1 = [0,1]/0 \sim 1$, and t is the coordinate of S^1-factor. We put

$$\Gamma_m = \bigcup_{(p,v_0,I)} (\Gamma_1(p_0, v_0, I))^m.$$

Let $L_m = SU(2)^m/SU(2)$, where $SU(2)$ acts by conjugation. Define a map

$$\tilde{\Phi}': \mathscr{A}_\ell(M) \times \Gamma_m \to \mathrm{Map}(D^2, SU(2)^m)$$

by

$$\tilde{\Phi}'(a, (\ell_1, \ldots, \ell_m))(x) = (h_{\ell_1(\cdot,x)}(a), \ldots, h_{\ell_m(\cdot,x)}(a)).$$

Now $\tilde{\Phi}'$ induces a map

$$\tilde{\Phi}': \mathscr{B}_\ell(M) \times \Gamma_m \to \mathrm{Map}(D^2, L_m).$$

Choose $(\beta_i)_{i \in Z_+}$ $(\beta_i > 0)$. We define a Banach space $C^\beta(L_m, \boldsymbol{R})$ of smooth functions, ψ on L_m such that the norm

$$\|\psi\|_\beta = \sum_{i=1}^{\infty} \beta_i \operatorname*{map}_{i \in L_m} |D^i \psi(x)|$$

is finite. Let $u: D^2 \to [0, \infty)$ be a function with

$$\int_{D^2} u(x)\,\mathrm{d}x = 1.$$

We define

$$\Phi: \mathscr{B}_\ell(M) \times \Gamma_m \times C^\beta(L_m, \boldsymbol{R}) \to \boldsymbol{R}$$

by

$$\Phi([a], (\ell_1, \ldots, \ell_m), \psi) = \int_{D^2} \psi(\Phi'([a], (\ell_1, \ldots, \ell_m))(x)) u(x)\,\mathrm{d}x.$$

For $v \in \Gamma_m \times C^\beta(L_m, \boldsymbol{R})$, we put $f_v([a]) = \Phi([a], v)$. We define an order $\prec$ on $\bigcup_m L_m$ by

$$(\ell_1, \ldots, \ell_m) \prec (\ell'_1, \ldots, \ell'_{m'}) \Leftrightarrow \{\ell_1, \ldots, \ell_m) \subseteq \{\ell'_1, \ldots, \ell'_{m'}\}.$$

We choose a sufficiently large m and an element $\lambda = (\ell_1, \ldots, \ell_m)$ sufficiently dense in M and sufficiently large with respect to the order $\prec$. For each $x \in D^2$, the map Φ' induces a map $\Phi'_x: \mathscr{B}_\ell(M) \times \Gamma_m \to L_m$. We consider a union

$$\bigcup_{x \in D^2} \Phi_x(R(M), \lambda) \subset L_m.$$

Let L' be the complement of a small neighborhood of this set and let Ψ be the set of elements in $C^\beta(L_m, \boldsymbol{R})$, whose support is contained in L'. For $\phi \in \Psi$, we have a map $f_\phi: \mathscr{B}_\ell(M) \to \boldsymbol{R}$. By definition the support of f_ψ is contained in a complement of a neighborhood of $R(M)$. Our perturbed ASD equation corresponding to ϕ is

$$F_A + \tilde{*} F_A - \mathrm{grad}_{a_t} f_\phi \wedge dt - * \mathrm{grad}_{a_t} f_\phi = 0. \tag{6.2}$$

Here $\tilde{*}$ is the Hodge $*$-operator for $M \times \boldsymbol{R}$ and $*$ is the Hodge $*$-operator for M, and we put

$A = a_t + b_t\,dt$. In order to prove that $\mathcal{M}_\phi(j, k)$ is a smooth manifold, it suffices to show that, for each ASD connection $A \in \mathcal{M}(a, b)$, the operator

$$\mathcal{D}_A : \Omega^1_\ell \to \Omega^2_\ell, \qquad \alpha \mapsto (d_A + * d_A)\alpha - \mathrm{Hess}_{a_t} f_\phi(u_t)$$

is surjective, if ϕ is contained in a dense subset of Ψ. Here

$$A = a_t + b_t\,dt$$

$$u = u_t + v_t\,dt.$$

Now, for sufficiently large $\lambda = (\ell_1, \ldots, \ell_m)$, we consider the direct product $\mathcal{B}_\ell(M) \times C^\beta(L_m, \boldsymbol{R})$. In a way similar to [13], we can prove that the subset

$$\mathcal{M}od = \{((A, a, b), \phi) \in \mathcal{B}_{\ell,\delta}(M \times \boldsymbol{R}; R_j, R_i) \times C^\beta(L_m, \boldsymbol{R}) \mid A \text{ satisfies } (6.2)\}$$

is a smooth manifold. Hence, again we can apply the Sard–Smale theorem to the projection $\mathcal{M}od \to C^\beta(L_m, \boldsymbol{R})$. It follows that if ϕ is contained in a residual subset of $C^\beta(L_m, \boldsymbol{R})$, then the set $\{(A, a, b) \in \mathcal{B}_{\ell,\delta}(M \times \boldsymbol{R}; R_j, R_i) \mid A$ satisfies (6.2) for $\phi\}$ is a smooth manifold and $\mathcal{D}_A$ is surjective for each A in this set.

Thus, we proved that $\mathcal{M}_\phi(j, k)$ is a smooth submanifold for each ϕ in a residual subset of the parameter space Ψ.

We next turn to the proof of the dimension formula

$$\dim \mathcal{M}_\phi(j, k) = j - k + \dim R_j - 1 \tag{6.3}$$

in Theorem 2.3. We use the notion of spectral flow due to Atiyah–Patodi–Singer for this purpose. For simplicity, we consider the case when we can take $\phi = 0$. Following [13] we put

$$\mathcal{L}^i_{\ell,\delta} = L^2_{\ell,\delta}(M \times \boldsymbol{R}; su(2) \otimes p^*(\Lambda^i M))$$

and consider the differential operator, $\mathcal{D}_A : \mathcal{L}^1_{\ell,\delta} \oplus \mathcal{L}^2_{\ell,\delta} \to \mathcal{L}^1_{\ell,\delta} \oplus \mathcal{L}^2_{\ell,\delta}$,

$$\mathcal{D}_A(u, \varphi) = \left(-\frac{\partial u}{\partial t} + (*_g d_{a_t} - \mathrm{Hess}_{a_t} f - b_t \wedge) u + d^*_{a_t}\varphi, \frac{\partial \varphi}{\partial t} + d_{a_t} u \right)$$

where $A = a_t + b_t \wedge dt$. In case $R(M)$ is discrete, the index of this operator with $\delta = 0$ gives the dimension of $\mathcal{M}(j, k)$, and is equal to the index of the spectral flow of the family of operators $(u, \varphi) \mapsto ((*_g d_{a_t} - \mathrm{Hess}_{a_t} f - b_t \wedge) u + d^*_{a_t}\varphi, d_{a_t} u)$. But in our case where $R(M)$ is not necessary discrete, the operator $\mathcal{D}_A$ is not Fredholm for $\delta = 0$. So we need to consider the case $\delta > 0$. Then as in [13] this corresponds to pushing the 0 eigenvalues of the operator $(u, \varphi) \mapsto ((*_g d_{a_t} - \mathrm{Hess}_{a_t} f - b_t \wedge) u + d^*_{a_t}\varphi, d_{a_t} u)$ in the positive direction when $t \to -\infty$ and in the negative direction when $t \to \infty$. Thus, in case R_j contains no reducible connections, we have

$$\dim \mathcal{M}(i, j) + \dim \mathcal{M}(j, k) = \dim \mathcal{M}(i, k) + \dim R_j. \tag{6.4}$$

A similar formula can be proved in case when R_j contains reducible (trivial) connections. It is straightforward to prove (6.3) using (6.4) and a similar formula in case R_j contains reducible connections. The proof of Theorem 2.3 is now complete.

7. DECAY ESTIMATE

In Sections 7–9, we study a compactification of our moduli space (and prove Theorem 2.5, etc.). Roughly speaking, to prove Theorem 2.5, we need to show the two statements.

One is to find a family of (bubbling) connections for each element of $\mathcal{M}(j,\ell)\times_{R_\ell}\mathcal{M}(\ell,k)$. The other is to show that every element of the end is obtained in that way.

The first statement is analogous to Taubes' construction. This part is already discussed in [20]. In the next section we modify his argument and apply it to our situation. The second statement is analogous to Uhlenbeck's decay estimate (see, for example, [12]). This part is also parallel to [13]. In this section we discuss the second point in a similar way to [13, Section 9]. It seems to the author that Mrowka knew them when he wrote [20]. But since he did not write it there we give these arguments here.) We begin with proving a lemma. Let k and ℓ be sufficiently large positive integers. For $u:[-T-1, T+1]\to\mathcal{A}(M):t\mapsto u_t$, we define its norm by

$$\|u\|_{L^2_{\ell,\mathrm{loc}}}=\sup_{|t|<T}\|u\|_{L^2_\ell(M\times(t-1,t+1))}.$$

LEMMA 7.1 (Elliptic regularity in temporal gauge). *For each Λ, and each sufficiently small positive number ε_0, there exists a constant $C(\Lambda,\varepsilon_0)$ such that if $c_0, a_t\in\mathcal{A}(M)$ satisfies*

$$\begin{cases}|c_0|_{C^k}<\Lambda\\ \|c_0-a_t\|_{L^2_\ell}<\varepsilon_0 \quad \text{for } t\in[-T-1,T+1]\\ \dfrac{\partial a_t}{\partial t}=*F_{a_t}\end{cases}$$

then for each $t\in[-T,T]$ we have

$$\|a_t-c_0\|_{L^2_\ell}\le C(T\|c_0-a_t\|_{L^2_0}+\|c_0-a_0\|_{L^2_\ell}).$$

Proof. Let $\varphi:[-T-1,T+1]\to[0,1]$ be a function such that

$$\varphi(t)=\begin{cases}1 & \text{on } [-T,T]\\ 0 & \text{on } \partial[-T-1,T+1].\end{cases}$$

We put

$$A=c_0+(a_t-c_0)\cdot\varphi.$$

We need the following sublemma (where ℓ' is an integer sufficiently larger than 1 but sufficiently smaller than ℓ).

SUBLEMMA 7.2. *There exists a gauge transformation g on M such that if one puts $g^*A=c_0+B$ then we have*

$$d^*_{c_0}B=0 \tag{7.2.1}$$

$$\|B\|_{L^2_{\ell',\mathrm{loc}}}<C\|c_0-a_t\|_{L^2_{0,\mathrm{loc}}} \tag{7.2.2}$$

$$g=1 \quad \text{on } \partial(M\times[T-1,T+1]). \tag{7.2.3}$$

Proof. The lemma is essentially the same as in (for example) [12]. We take u_1 with

$$\begin{cases}d^*_{c_0}d_{c_0}u_1 & =d^*_{c_0}(A-c_0)\varphi\\ u_1 & =0 \quad \text{on } \partial(M\times[-T-1,T+1])\\ u_1 & \perp \operatorname{Ker} d_{c_0}.\end{cases}$$

Then we have

$$\|u_1\|_{L^2_{\ell'+1,\mathrm{loc}}}\le C\|a_t-c_0\|_{L^2_{\ell',\mathrm{loc}}}\le C\|a_t-c_0\|_{L^2_{0,\mathrm{loc}}}.$$

We put $\exp(-u_1) = g_1$. Then (7.3) implies

$$\| d_{c_0}^*(g_1^* A - c_0) \|_{L^2_{\ell', \mathrm{loc}}} \le C \| (a_t - c_0)^2 \|_{L^2_{\ell', \mathrm{loc}}} \le C \| (a_t - c_0)^2 \|_{L^2_{1, \mathrm{loc}}}$$
$$\le C \| (a_t - c_0) \|^2_{L^2_{0,\delta}}.$$

We next take u_2 such that

$$\begin{cases} d_{c_0}^* d_{c_0} u_2 & = d_{c_0}^*(g_1^* A - c_0)\varphi \\ u_2 & = 0 \quad \text{on } \partial(M \times [T-1, T+1]) \\ u_2 & \perp \operatorname{Ker} d_{c_0}. \end{cases} \tag{7.4}$$

Then we have

$$\| u_2 \|_{L^2_{\ell'+1, \mathrm{loc}}} \le C \| a_t - c_0 \|^2_{L^2_{0, \mathrm{loc}}}. \tag{7.5}$$

Hence by putting $\exp(-u_2) = g_2$, we obtain

$$\| d_{c_0}^*(g_1^* g_2^* A) \|_{L^2_{\ell', \mathrm{loc}}} \le C \| a_t - c_0 \|^3_{L^2_{0, \mathrm{loc}}}. \tag{7.6}$$

We define g_i inductively in the same way. Equations (7.5) and (7.6) and similar formulas for g_i imply that the sequence

$$\lim_{i \to \infty} g_i g_{i-1} \cdots g_2 g_1$$

converges in $L^2_{\ell'+1, \mathrm{loc}}$-norm. Let g be its limit. It is easy to see that

$$\begin{cases} \| g - id \|_{L^2_{\ell'+1, \mathrm{loc}}} & \le C \| a_t - c_0 \|_{L^2_{0, \mathrm{loc}}} \\ d_{c_0}^*(g^* A) & = 0. \end{cases}$$

The proof of Sublemma 7.2 is now complete.

We go back to the proof of Lemma 7.1. Let B be as in Sublemma 7.2. Then, since

$$d_{c_0}^* B = 0, \qquad F^+_{c_0 + B} = 0$$

the standard elliptic regularity implies that

$$\| B \|_{L^2_{\ell + 100, \mathrm{loc}}} \le C \| B \|_{L^2_{\ell', \mathrm{loc}}} \le C \| a_t - c_0 \|_{L^2_{0, \mathrm{loc}}}. \tag{7.7}$$

We take a gauge transformation g' such that

$$\begin{cases} g'(x, o) = 1 \\ g'^*(c_0 + B) \text{ has no } dt \text{ components.} \end{cases}$$

Equation (7.7) implies

$$\| g' - id \|_{L^2_{\ell + 99}(M \times (t-1, t+1))} \le C(1 + |t|) \| a_t - c_0 \|_{L^2_{0, \mathrm{loc}}}. \tag{7.8}$$

We put $h = g'g$ and $h^*(d + a_t) = d + b_t$. By (7.2) and (7.8) we have

$$\| b_t - c_0 \|_{L^2_{\ell + 98}(M \times (t-1, t+1))} \le C(1 + |t|) \| a_t - c_0 \|_{L^2_{0, \mathrm{loc}}}. \tag{7.9}$$

Since both $d + a_t$ and $h^*(d + a_t)$ do not have a dt component it follows that

$$\frac{dh}{dt} = 0. \tag{7.10}$$

Equations (7.9) and (7.10) imply that

$$\begin{aligned}\|h - id\|_{L^2_{\ell+1}} &\le C\|a_0 - b_0\|_{L^2_\ell(M)}\\ &\le C(\|a_0 - c_0\|_{L^2_\ell(M)} + \|b_0 - c_0\|_{L^2_\ell(M)})\\ &\le C(\|a_0 - c_0\|_{L^2_\ell(M)} + \|a_t - c_0\|_{L^2_\ell(M)}).\end{aligned}$$

On the other hand, we have $a_t = (h^{-1})^* b_t$, $\|b_t - c_0\|_{L^2_\ell(M)} < C\|a_t - c_0\|_{L^2_\ell(M)}$. Hence, we conclude

$$\|a_t - c_0\|_{L^2_\ell(M)} \le C(\|a_0 - c_0\|_{L^2_\ell(M)} + T\|a_t - c_0\|_{L^2_\ell(M)}).$$

The proof of Lemma 7.1 is now complete.

For the next lemma, we assume Condition 2.1 (but we do not assume that M is a homology sphere).

LEMMA 7.11 (Gauge fixing on M). *There exists a positive number ε such that if $[c] \in R(M)$, $a \in \mathcal{A}(M)$, $\|c - a\|_{L^2_\ell} < \varepsilon$, then there exists $[c'] \in R(M)$ and $u \in L^2_{\ell+1}(M; su(2))$, such that*

$$\|c - c'\|_{L^2_\ell} < C\varepsilon \tag{7.11.1}$$

$$c' - (\exp u)^* a \in (\operatorname{Ker} d_{c'})^\perp. \tag{7.11.2}$$

Proof. By the compactness of $R(M)$ there exist C_1 and ε_1 such that if $[c] \in R(M)$, $b' \in T_{[c]}R(M)$, $d_c b' = d_c^* b' = 0$, $\|b'\|_{L^2} < \varepsilon < \varepsilon_1$, then there exists $[c'] \in R(M)$ satisfying

$$\|c' - (c + b')\|_{L^2_\ell} < C_1 \varepsilon^2. \tag{7.12}$$

Now let $b = a - c$. We decompose b as $b = b_1^{(1)} + b_2^{(1)} + b_3^{(1)}$ such that

$$\begin{cases} b_1^{(1)} \in \operatorname{Im} d_c \\ b_2^{(1)} \in \operatorname{Ker} d_c \cap \operatorname{Ker} d_c^* \\ b_3^{(1)} \in (\operatorname{Ker} d_c)^\perp. \end{cases}$$

By assumption $\|b_i^{(1)}\|_{L^2_\ell} < C\varepsilon$. Using (7.12) we find $[c_1'] \in R(M)$ such that

$$\|c_1' - (c + b_2^{(1)})\|_{L^2_\ell} < C\varepsilon^2.$$

We take $u_1 \in L^2_\ell(M; su(2))$ with

$$\begin{cases} u_1 & \perp \operatorname{Ker} d_c \\ d_c u_1 & = b_1^{(1)} \end{cases}$$

and put $g_1 = \exp(-u_1)$. Then

$$\|c_1' - (g_1^* a + b_3^{(1)})\|_{L^2_\ell} < C'\varepsilon^3.$$

We next decompose $c_1' - g_1^* a$ and continue in the same way. Then $\lim_{n\to\infty} g_n \cdots g_1$ converges to g in $L^2_{\ell+1}$-norm, and we have

$$\begin{cases} \|g - id\|_{L^2_{\ell+1}} < C\varepsilon \\ \lim_{n\to\infty} c_n' = c' \quad \text{in } L^2_\ell \\ \|c' - c\|_{L^2_\ell} < C\varepsilon \\ c' - g^* a \in (\operatorname{Ker} d_c)^\perp \\ [c'] \in R(M). \end{cases}$$

The proof of Lemma 7.11 is now complete.

Next we prove the following analogy of [13, Lemma 9.1].

LEMMA 7.13. *Let M be a manifold satisfying* (2.1). *Then there exist ε, λ and C such that if $d + a_t$ is an $su(2)$-connection on $M \times [-T, T]$ without dt-component, $[c] \in R(M)$, and if*

$$|a_t - c|_{L^2_\ell} < \varepsilon \tag{7.14.3}$$

$$\frac{\partial a_t}{\partial t} = * F_{a_t} \tag{7.14.4}$$

then there exists a gauge transformation $g \in \mathscr{G}_{\ell+1}(M)$ such that

$$|g^* a_t - c|_{L^2_\ell} \leq C e^{-\lambda \beta_T(t)}. \tag{7.15}$$

Here $\beta_T(t) = \inf\{T - t, T + t\}$.

Proof. The proof is a combination of the argument of the proof of [13, Lemma 9.1] and the lemmas we gave in this section. (In fact, the argument of the present paper is also necessary in the situation of [13]. What the author quoted as the "standard elliptic estimate", in [13] just before Sublemma 9.8, should be Lemma 7.1 of this section.) (In order to simplify the notation, in this section we do not consider the perturbation of the equation. The argument to deal with the perturbation is the same as [13].) First by using Lemma 7.11, we may assume

$$a_0 - c \in (\operatorname{Ker} d_c)^{\perp}.$$

We put $u(t) = a_t - c$. Decompose it as $u(t) = \alpha(t) + \beta(t)$ with

$$\begin{cases} d_c^* \alpha(t) = 0 \\ \beta(t) = \operatorname{Im} d_c. \end{cases}$$

Then we have

$$\|\alpha(t)\|_{L^2_{\ell,\delta}} < C\varepsilon, \qquad \|\beta(t)\|_{L^2_{\ell,\delta}} < C\varepsilon \tag{7.16.1}$$

$$\frac{\partial \alpha(t)}{\partial t} = * d_c \alpha(t) - E_1(\alpha(t), \beta(t)) \tag{7.16.2}$$

$$\frac{\partial \beta(t)}{\partial t} = E_2(\alpha(t), \beta(t)) \tag{7.16.3}$$

$$\alpha(0) = 0 \tag{7.16.4}$$

with

$$\|E_i(\alpha(t), \beta(t))\|_{L^2_\ell} < C(\|\alpha(t)\|\,\|_{L^2_\ell} + \|\beta(t)\|_{L^2_\ell})^2. \tag{7.17}$$

We decompose

$$\alpha(t) = \alpha_+(t) + \alpha_-(t) + \alpha_0(t)$$

such that $\alpha_+(t)$, $\alpha_-(t)$, $\alpha_0(t)$ belong to the subspace spanned by positive, negative, and zero eigenvectors of $*d_c$, respectively. We put $g_\pm(t) = \|\alpha_\pm(t)\|_{L^2(M \times [t-1, t+1])}$, $h(t) = \|\alpha_0(t) + \beta(t)\|_{L^2(M \times [t-1, t+1])}$ and $\hat{h}(t) = \|\alpha_0(t) + \beta(t)\|_{L^2_\ell(M \times [t-1, t+1])}$, $\hat{g}_\pm(t) = \|\alpha_\pm(t)\|_{L^2_\ell(M \times [t-1, t+1])}$. We have $h(0) = \hat{h}(0) = 0$. Then by (7.16) and (7.17) we have

$$\frac{dg_+}{dt} \geq \lambda g_+ - C(g_- + h)(\hat{g}_- + \hat{h}) \tag{7.18.1}$$

$$\frac{dg_-}{dt} \le -\lambda g_- + C(g_+ + h)(\hat{g}_+ + \hat{h}) \tag{7.18.2}$$

$$\left|\frac{dh}{dt}\right| \le C(g_+ + g_- + h)(\hat{g}_+ + \hat{g}_- + \hat{h}). \tag{7.18.3}$$

On the other hand, by Lemma 7.1 we have

$$\begin{cases} \hat{h}(t) \le C\left(t\hat{h}(0) + \sup\limits_{|s|<t} h(s)\right) \\ \hat{g}_\pm(t) \le C\left(t\hat{g}_\pm(0) + \sup\limits_{|s|<t} g_\pm(s)\right). \end{cases} \tag{7.19}$$

Now we can prove the following.

SUBLEMMA 7.20. *There exists $\mu_0 > 0$ such that if $T^2\delta < \mu_0$, $T\varepsilon_0 < \mu_0$ then* (7.18), (7.19) *and $\hat{h}(0)$, $\hat{g}_\pm(0) < \delta$, $\hat{h}(t)$, $\hat{g}_\pm(t) < \varepsilon_0$ imply*

$$\begin{cases} g_+(t) \le C(\mathrm{e}^{-\lambda\beta_T(t)} + \delta) \\ g_-(t) \le C(\mathrm{e}^{-\lambda\beta_T(t)} + \delta) \\ h(t) \le C(\mathrm{e}^{-\lambda\beta_T(t)} + \delta). \end{cases}$$

Proof. As in [13, Sublemma 9.8], we prove

$$|h| \le C(\varepsilon^n + \varepsilon \mathrm{e}^{-\lambda\beta_T(t)} + \delta) \tag{7.21.2n}$$

$$|g_\pm| \le C(\varepsilon^n + \varepsilon \mathrm{e}^{-\lambda\beta_T(t)} + \delta) \tag{7.22.2n. \pm}$$

by an induction on n (n is a half integer). We assume (7.21.2n), (7.22.2n − 1. ±). We put

$$\tilde{g}_+(t) = \mathrm{e}^{-\lambda(t-t_0)} g_+(t).$$

Then by (7.18) and (7.19), we have

$$\begin{aligned} \varepsilon \mathrm{e}^{-\lambda(T-t_0)} &\ge \tilde{g}_+(T) \\ &\ge g_+(t_0) - \int_{t_0}^{T} C\mathrm{e}^{-\lambda(t-t_0)}(\varepsilon^n + \varepsilon\mathrm{e}^{-\lambda\beta_T(t)} + \delta)(T(\varepsilon^n + \varepsilon\mathrm{e}^{-\lambda\beta_T(0)} + \delta) \\ &\quad + \varepsilon^n + \varepsilon\mathrm{e}^{-\lambda\beta_T(t)} + \delta)\,\mathrm{d}t. \end{aligned}$$

Then (7.22.2n. +) follows. The other part of the induction is similar to the argument of [13, Section 9] and is omitted. (We remark that we need to use Lemma 7.11 again to replace c several times.) The proof of Lemma 7.13 is now complete.

Now combining Lemmas 7.1, 7.11 and 7.13 in a way similar to [13, Section 9], we obtain the following.

LEMMA 7.23. *There exists $T_0, \varepsilon, \lambda$ such that if $d + a_t$ be an $su(2)$-connection on $M \times [-T, T]$ without dt component, and if*

$$T > T_0 \tag{7.24.1}$$

$$\frac{\partial a_t}{\partial t} = *F_{a_t} \tag{7.24.2}$$

$$\sup_t \left\|\frac{\partial a_t}{\partial t}\right\|_{L^2_{\ell,\delta}} < \varepsilon \tag{7.24.3}$$

then there exists $[c] \in R(M)$ *and* $g \in \mathscr{G}_{\ell+1}(M)$ *such that*

$$\|g^* a_t - c\|_{L^2_\ell} \le C e^{-\lambda \beta_T(t)}. \tag{7.25}$$

Thus we have established the decay estimate we need in later sections.

8. TAUBES' CONSTRUCTION

In [20], Mrowka generalized Taubes' (and other people's) argument to the case when (in our terminology) the set $R(M)$ is not necessarily discrete. In this section we first modify and apply his argument to our situation. We give a self-contained proof rather than quoting Mrowka since the function space we use is a bit different from Mrowka's, our situation is different, and since Mrowka's proof is available only in his thesis so far. We consider only the situation of Theorem 6.1 (the case of 4-manifold N with boundary M and a generic metric g on it.) The case of $N = M \times \boldsymbol{R}$ and perturbed ASD equation are similar.

Let N_1 be a manifold satisfying the conditions for N in Theorem 6.1, and N_2 be another Riemannian 4-manifold such that N_2 minus a compact set is isometric to $M \times (-\infty, 0)$. We assume that the moduli spaces of ASD connections on N_1, N_2 satisfy the conclusion of Theorem 6.1. Then we have boundary maps

$$\Pi^r_{i,N_1} : \mathscr{M}(N_1; R_i) \to R_i(M)$$

$$\Pi^\ell_{i,N_2} : \mathscr{M}(N_2; R_i) \to R_i(M).$$

In a way similar to the proof of Theorem 6.1, we can prove that, for generic metrics on N_1, N_2 these two maps are transversal to each other. We patch $N_1 - M \times (T, \infty)$ and $N_2 - M \times (-\infty, -T)$ along $M \times \{T\} = M \times \{-T\}$ and obtain $N(T)$.

PROPOSITION 8.1. *For compact subsets* $K_j \in \mathscr{M}(N_j; R_i)$, $j = 1, 2$, *there exists* T_0 *such that there exists a smooth embedding*

$$\mathscr{P}a\ell : K_1 \times_{R_i(M)} K_2 \times (T_0, \infty) \to \bigcup_{T > T_0} \mathscr{M}(N(T)).$$

Proof. Let $\chi_j, \chi'_{j,\pm}, \chi''_{j,\pm} : N(T) \to [0, \infty)$, $\chi_{j,\pm} : N_j \to (0, \infty)$ be maps such that

$$\chi'_{1,\pm}(x) \begin{cases} = 1 & \text{if } x = (y, t),\ t < \pm \tfrac{1}{2}T - 1 \text{ or } x \in N_1 - M \times (0, \infty) \\ = 0 & \text{if } x = (y, t),\ t > \pm \tfrac{1}{2}T + 1 \text{ or } x \in N_2 - M \times (-\infty, 0) \\ \in [0, 1] & \text{always} \end{cases}$$

$$\chi'_{2,\pm}(x) = 1 - \chi'_{1,\pm}(x)$$

$$\chi''_1(x) \begin{cases} = 1 & \text{if } x = (y, t),\ t < -S \text{ or } x \in N_1 - M \times (0, \infty) \\ = 0 & \text{if } x = (y, t),\ t > +S \text{ or } x \in N_2 - M \times (-\infty, 0) \\ \in [0, 1] & \text{always} \end{cases}$$

$$\chi''_2(x) = 1 - \chi''_1(x).$$

Here S is a number which is sufficiently large but is sufficiently smaller than T. We fix S later:

$$\chi_{1,\pm}(x) \begin{cases} = 1 & \text{if } x \in N_1 - M \times (T \pm \tfrac{1}{2}T - 1, \infty) \\ = 0 & \text{if } x = (y, t),\ t > T \pm \tfrac{1}{2}T + 1 \\ \in [0, 1] & \text{always} \end{cases}$$

$$\chi_{2,\pm}(x)\begin{cases} = 0 & \text{if } x\in N_2 - M\times(-\infty,\ -(T\mp\frac{1}{2}T+1)) \\ = 1 & \text{if } x=(y,t),\ t> -(T\mp\frac{1}{2}T-1) \\ \in[0,1] & \text{always} \end{cases}$$

$$\chi_1(x)\begin{cases} = 1 & \text{if } x\in N_1 - M\times(0,\infty) \text{ or } x=(y,t),\ t< -\frac{3}{4}T-1, \\ = 0 & \text{if } x\in N_2 - M\times(-\infty,0) \text{ or } x=(y,t),\ t< -\frac{3}{4}T+1, \\ \in[0,1] & \text{always} \end{cases}$$

$$\chi_2(x)\begin{cases} = 1 & \text{if } x\in N_2 - M\times(-\infty,0) \text{ or } x=(y,t),\ t> +\frac{3}{4}T+1 \\ = 0 & \text{if } x\in N_1 - M\times(0,\infty) \text{ or } x=(y,t),\ t< +\frac{3}{4}T-1 \\ \in[0,1] & \text{always.} \end{cases}$$

Here and hereafter we regard

$$\begin{aligned} N(T) &= (N_1-(M\times(T,\ \infty)))\cup(N_2-(M\times(-\infty,\ -T))) \\ &= (N_1-(M\times(0,\ \infty)))\cup(N_2-(M\times(-\infty,0)))\cup M\times(-T,T) \\ &= (N_1-(M\times(2T,\ \infty)))\cup(N_2-(M\times(-\infty,0))) \\ &= (N_1-(M\times(0,\ \infty)))\cup(N_2-(M\times(-\infty,\ -2T))). \end{aligned}$$

Now for $[A_j,a]\in\mathcal{M}(M_j,R_i)$, with $a\in R_i(M)$, we define $A'(T)\in\mathcal{M}(N(T))$ by

$$A'(T)(x)=\begin{cases} A_1(x) & \text{if } x\in N_1-M\times(0,\infty) \text{ or } x=(y,t)\ t<-T+1 \\ A_2(x) & \text{if } x\in N_2-M\times(-\infty,0) \text{ or } x=(y,t)\ t>T-1 \\ \chi_1(x)A_1(x)+\chi_2(x)A_2(x) \\ \quad +(1-\chi_1(x)-\chi_2(x))a & \text{if } x\in M\times(-T,T). \end{cases}$$

Using the decay estimate of the last section, it is easy to see that this connection is almost ASD. Our purpose is to find an ASD connection in a neighborhood of this connection. The fundamental idea by Taubes is that one can do it if the second cohomology of the Atiyah–Hitchin–Singer complex for $A'(T)$ vanishes. Let $\Omega^i_{\ell,\delta}(N_j)$, $\Omega^i_\ell(N(T))$ be as in Section 6. (Since $N(T)$ is compact, we need not perturb the norm using ℓ, δ.) We consider the operator

$$d^+_{A'(T)}\circ(d^+_{A'(T)})^*:\Omega^2_\ell(N(T))\to\Omega_\ell(N(T)). \tag{8.3}$$

We remark that Assumption 2.1 implies that the similar operators

$$d^+_{A_j}\circ(d^+_{A_j})^*:\Omega^2_{\ell,\delta}(N_j)\to\Omega^2_{\ell,\delta}(N_j)$$

are surjective. The trouble here is that the first eigenvector of map (8.3) is *not* bounded away from 0 as T goes to infinity. So we need to change the norm. Let $e_\delta:N(T)\to[1,\infty)$ be a smooth function such that

$$\begin{cases} |e_\delta(x,t)-e^{\delta|t+T|}|<1, & t\le 0 \\ |e_\delta(x,t)-e^{\delta|t-T|}|<1, & t\ge 0 \end{cases} \tag{8.4}$$

and define the $\|\ \|^2_{\ell,\delta}$ norm on $\Omega^i(N(T))$ by

$$\|u\|^2_{\ell,\delta}=\int e_\delta\sum_{j=0}^{\ell}|\nabla^j u|^2\,dx.$$

Next we define $\chi : N(T) \to [0, 1]$ by

$$\chi(x) = \begin{cases} 1 & \text{if } x = (y, t),\ t \in [-T+2, T-2] \\ 0 & \text{if } x = (y, t),\ t \notin [-T+1, T-1] \quad \text{or } x \in N_1 - M \times (0, \infty) \\ & \text{or } x \in N_2 - M \times (-\infty, 0) \end{cases}$$

and put

$$\Omega^1_{\ell,\delta,a} = \left\{ (B, b) \,\middle|\, \begin{array}{l} A \in \Omega^1_{\ell,\delta}(N(T)) \\ b \in \mathscr{A}(M) \\ d_a b = 0 \end{array} \right\}$$

$$\|(B, b)\|^2_{\ell,\delta,a} = \|b\|^2_{\ell,\delta} + \|B - \chi b\|^2_{\ell,\delta}.$$

LEMMA 8.5. *There exists a positive number C independent of T such that for each $u \in \Omega^2_{\ell,\delta}(N(T))$ there exists $(B, b) \in \Omega^1_{\ell,\delta,a}$ with the following properties:*

$$d^+_{A(T)} B = u, \tag{8.5.1}$$

$$\|(B, b)\|_{\ell+1,\delta,a} \leq C \|u\|_{\ell,\delta}. \tag{8.5.2}$$

Moreover, the map $u \mapsto (B, b)$ is gauge invariant and smooth.

Proof. We prove the lemma by a kind of alternative method. We put $u_1 = \chi''_1 u$, $u_2 = \chi''_2 u$. The sections u_1, u_2 have their supports on $N_1 - M \times (2T, \infty)$ and $N_2 - M \times (-\infty, -2T)$, respectively. Hence, we regard them as elements of $\Omega^2_{\ell,\delta}(N_j)$, $(j = 1, 2)$, respectively. By definition, we have

$$\|u_j\|_{\ell,\delta} \leq C \|u\|_{\ell,\delta}.$$

Now the basic idea of the proof of Lemma 8.5 is the following.

SUBLEMMA 8.6. (cf. [20]). *For each $(u_1, u_2) \in \Omega^2_{\ell,\delta}(N_1) \times \Omega^2_{\ell,\delta}(N_2)$ there exist $(B_1, b) \in \mathscr{H}^1_{\ell+1,\delta}(N_1, a)$ and $(B_2, b) \in \mathscr{H}^1_{\ell+1,\delta}(N_2, a)$ such that*

$$d_{A_j} B_j = u_j \tag{8.6.1}$$

$$\|(B_j, b)\|_{\ell+1,\delta} \leq C(\|u_1\|_{\ell,\delta} + \|u_2\|_{\ell,\delta}). \tag{8.6.2}$$

We remark here that the second component b coincides for $j = 1, 2$.

Proof. We consider a commutative diagram as shown in Fig. 6.

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & \Omega^2_{\ell,\delta}(N_1) \oplus \Omega^2_{\ell,\delta}(N_2) \arrow[r] \arrow[u] & \Omega^2_{\ell,\delta}(N_1) \oplus \Omega^2_{\ell,\delta}(N_2) \arrow[r] \arrow[u] & 0 & \\
0 \arrow[r] & H \arrow[r] \arrow[u] & \mathscr{H}^1_{\ell,\delta}(N_1, a) \oplus \mathscr{H}^1_{\ell,\delta}(N_2, a) \arrow[r] \arrow[u, "d^+_{A_1} \oplus d^+_{A_2}"'] & T_{[a]} R(M) \arrow[r] \arrow[u] & 0 \\
 & & T_{[A_1]} \mathscr{M}(N_1) \oplus T_{[A_2]} \mathscr{M}(N_2) \arrow[r] \arrow[u] & T_{[a]} R(M) \arrow[r] \arrow[u] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Fig. 6.

Here H is defined such that the third horizontal sequence is exact. By Assumption 2.1 the third vertical sequence is exact. We remark that the map in the fourth horizontal line is given by $\Pi^r_* + \Pi^\ell_*$. Hence, our transversality assumption implies that it is exact. Then a simple diagram chaise implies that the second vertical line is exact. In other words, we can find $(B_1, b), (B_2, b)$ satisfying (8.6.1). Then (8.6.2) is a consequence of open mapping theorem.

We go back to the proof of Lemma 8.5. Let $(B_1, b), (B_2, b)$ be as in Sublemma 8.6. We put $b^{(1)} = b$:

$$B^{(1)}(x) = \begin{cases} B_1^{(1)}(x) & \text{if } x \in N_1 - M \times (0, \infty), \\ B_2^{(1)}(x) & \text{if } x \in N_2 - M \times (-\infty, 0), \\ \chi_{1,+}(x) B_1^{(1)}(x) + \chi_{2,-}(x) B_2^{(1)}(x) & \text{if } x \in M \times (-T, T). \\ \quad + (1 - \chi_{1,+}(x) - \chi_{2,-}(x)) b^{(1)} & \end{cases}$$

We remark that the support of the section $u - d_a^+ B^{(1)}$ is contained in the set $(M \times (-\frac{1}{2}T - 1, -\frac{1}{2}T + 1)) \cup (M \times (\frac{1}{2}T - 1, \frac{1}{2}T + 1))$. Then we have

$$\| B^{(1)} \|_{\ell+1,\delta} < C \| u \|_{\ell,\delta} \tag{8.7.1}$$

$$\| u - d_A^+ B^{(1)} \|_{\ell,\delta} < C e^{-T/2} \| u \|_{\ell,\delta}. \tag{8.7.2}$$

We next replace u by $u - d_a^+ B^{(1)}$ to obtain $(B_j^{(2)}, b^{(2)})$. Then formula (8.7) and similar formulas imply that $(\sum_{i=1}^{\infty} B_j^{(i)}, \sum_{i=1}^{\infty} b^{(i)})$ converges to an element (B_j, b) of $\mathscr{H}^1_{\ell+1,\delta}(N_j, a)$ satisfying (8.5.1) if T is sufficiently large. Equation (8.7.2) and similar formulas imply that (B_j, b) satisfies (8.5.2). We remark that the constant in (8.5.2) is uniformly bounded when A_1, A_2 moves in a compact subset. By construction the connection (B_j, b) depends smoothly on A_j, T and is gauge invariant. The proof of Lemma 8.5 is now complete.

We now go back to the proof of Proposition 8.1. First we remark that Lemma 7.23 implies the estimate

$$\| F^+_{A(T)} \|_{\ell,\delta,a} < C e^{-\delta T}, \tag{8.8}$$

provided we choose δ sufficiently small compared to the constant λ in Lemma 7.23. We regard $\chi_j'' \cdot F^+_{A(T)}$ as sections on N_j and put $u_j^{(1)} = \chi_j'' \cdot F^+_{A(T)}$ $(j = 1, 2)$. Then we apply Lemma 8.5 to obtain $(B_j^{(1)}, b^{(1)})$. Let χ_j''' be a cut function on N_j such that

$$\chi_1'''(x) \begin{cases} = 1 & \text{if } x \in N_1 - M \times (0, \infty) \text{ or } x = (y, t),\ t < T - S \\ = 0 & \text{if } x = (y, t),\ t < T + S \\ \in [0, 1] & \text{always} \end{cases}$$

$$\chi_2'''(x) \begin{cases} = 1 & \text{if } x \in N_2 - M \times (-\infty, 0) \text{ or } x = (y, t),\ t > -T + S \\ = 0 & \text{if } x = (y, t),\ t < -T - S \\ \in [0, 1] & \text{always.} \end{cases}$$

We regard $\chi_j''' \cdot B_j^{(1)}$ as $su(2)$-valued 1-forms on $N(T)$ and put $A^{(1)} = \chi_1''' \cdot B_1^{(1)} + \chi_2''' \cdot B_2^{(1)}$. Then we have

$$\| F^+_{A^{(1)}} \|_{\ell,\delta,a+b^{(1)}} < C\left((e^{-\delta T})^2 + \frac{1}{S} \right). \tag{8.9}$$

We repeat the same procedure to obtain $A^{(j)}$. Then $\lim_{j \to \infty} A^{(j)}$ converges in $L^2_{\ell,\mathrm{loc}}$ topology to an ASD connection $\mathscr{P}a\ell(A_1, A_2, T)$. Furthermore, we can prove easily that the map $(A_1, A_2) \mapsto \mathscr{P}a\ell(A_1, A_2, T)$ is C^1-close to the identity map (if we put $L^2_{\ell,\mathrm{loc}}$-norm in the target space.) On $\Omega^2_\ell(N(T))$ we define a norm $\| \ \|_{\ell;1,2}$ by

$$\|u\|_{\ell;1,2} = \|u\|_{L^2_\ell(N(T))} + \|u\|_{L^1_\ell(M\times[-T,T])}\,.$$

LEMMA 8.10. *For each* $v \in \Omega^1_\ell(N(T))$ *and* $w_j \in T_{[A_j]}(\mathcal{M}(N_i))$, *we have*

$$\|\mathcal{P}at_*(w_1, w_2) - d_{\mathcal{P}at(A_1,A_2)}(u)\|_{\ell;1,2} > \varepsilon \cdot \|\mathcal{P}at_*(w_1, w_2) - d_{\mathcal{P}at(A_1,A_2)}(u)\|_{\ell;1,2}$$

for $T > T_0$. *Here* ε *is a positive number independent of* T.

The proof is omitted. In fact, the proof is much simpler than that of [13], since here the connection on $M\times[-T,T]$ is irreducible.

Now combining formula 8.9 and Lemma 8.10 we can prove that the map is an embedding. (The argument to prove this is exactly the same as in [13] and is omitted.) See also the proof of Proposition 9.3.

9. COMPACTIFICATION OF MODULI SPACES

In this section we combine the results of the last two sections and study the compactification of the moduli spaces of ASD connections. The argument is parallel to one in [13] and hence we describe only the outline. First we recall that we studied the case of 4-manifold with boundary and with generic metric on it, in last section. But the results can be generalized to the case of $M\times\boldsymbol{R}$ and $\boldsymbol{R}$-invariant perturbation by a minor change (see, for example, [13]). We state the conclusion here. First we consider the case when M is a homology sphere.

PROPOSITION 9.1. *Let* $A \subseteq R_i$ *and* $B \subseteq R_k$ *be open subsets. Define* $\mathcal{M}(M; A, R_j)$, $\mathcal{M}(M; R_j, B)$ *as in Section 2. For compact subsets* $K_1 \in \mathcal{M}(M; A, R_j)$, $K_2 \in \mathcal{M}(M; R_j, B)$ *there exists* T_0 *such that there exists a smooth embedding*

$$\mathcal{P}at\colon K_1 \times_{R_i(M)} K_2 \times (T_0, \infty) \to \mathcal{M}(M; R_i, R_k),$$

provided a transversality condition for boundary maps is satisfied.

Hereafter in this section, we always assume appropriate transversality conditions for moduli spaces and boundary maps without mentioning it. Now we need to define the following (cf. [8, 13]).

Definition 9.2. Let $K_1 \in \mathcal{M}(N; R_j)$, $K_2 \in \mathcal{M}(M; R_j, B)$ be compact subsets and $\varepsilon, T, C > 0$. We say that $A \in \mathcal{M}(N; B)$ is a *standard model of type* $(K_1, K_2; T, C, \varepsilon)$ if there exists $[A_j] \in K_j$, $S > T$, and $[A'] = [A]$, $[c] \in R_j$, $[c'] \in B$ with the following properties.

Define embeddings $I_1\colon N - M\times[0,\infty) \to N$, $I_2\colon M\times[-T,T] \to N$ by $I_1(x) = x$, $I_2(x,t) = (x, t+S)$. Then we have

$$\|I_j^*(A') - A_j\|_{C^\ell} < \varepsilon \tag{9.2.1}$$

$$\|A' - c\|_{C^k}(x,t) < C\exp(-d(t,\{0, S-T\})/C) \quad \text{on } M\times[0, S-T] \tag{9.2.2}$$

$$\|A' - c'\|_{C^k}(x,t) < C\exp(-d(t, S+T)/C) \quad \text{on } M\times[S+T, \infty). \tag{9.2.3}$$

Now an application of the results of last section and the argument in [12] (see also [13]) is the following:

PROPOSITION 9.3. *For each compact $K_1 \in \mathcal{M}(N; R_j)$, $K_2 \in \mathcal{M}(M; R_j, B)$ there exist $\varepsilon, T, C > 0$ such that if $A \in \mathcal{M}(N; B)$ is a standard model of type $(K_1, K_2; T, C, \varepsilon)$ then A is gauge equivalent to an image of the map $\mathcal{P}at$ (which is defined in the same way as $\mathcal{P}at$ in last section).*

Proof (sketch). Let $[A_j] \in K_j$ be as in Definition 9.2. We may replace A by A'. Then by (9.2.2) and the construction of the map $\mathcal{P}at$ we have

$$\| A' - \mathcal{P}at(A_1, A_2; S) \|_{C^k}(x, t) < C \exp(-d(t, \{0\} \cup \{S - T, S + T\})/C).$$

We join A' and $\mathcal{P}at(A_1\ A_2; S)$ by a path in the set of connections on N. Then using the construction of the last section again we can project this path to a short path in the set of ASD connections. It induces a path ℓ in $\mathcal{M}(N; B)$. On the other hand, the dimension counting (based on the Index theorem) and the construction of $\mathcal{P}at$ imply $\mathcal{P}at$ is of maximal rank. Hence, local contractibility of $\mathcal{M}(N; B)$ implies that $\mathcal{P}at$ is a diffeomorphism on a set with fixed small size. It follows that the path ℓ is lifted to $K_1 \times K_2 \times [T_0, \infty)$ (since one side of its boundary, $\mathcal{P}at(A_1\ A_2; S)$, is lifted and the path is short). Therefore, the other end A' is also contained in the image of $\mathcal{P}at$. The proof is now complete.

In a similar way we can define standard models in case $K_\ell \in \mathcal{M}(M; R_{j_\ell}, R_{j_{\ell+1}})$, $\ell = 2, 3, \ldots, n$ (the case $n = 2$ is Definition 9.2). A proposition similar to 9.3 holds for them.
Next we remark that the decay estimate of Section 7 implies the following.

COROLLARY 9.4. *For each compact subsets $K_1 \in \mathcal{M}(N; R_j)$, $K_2 \in \mathcal{M}(M; R_j, B)$ and $\varepsilon, T, C > 0$, there exists $\varepsilon', T' > 0$ with the following properties. Define $I_2' : M \times [-T', T'] \to N$ as in Definition 9.2 using $S' > T'$.*
Suppose that there exists $[A_j] \in K_j$, $[A'] = [A]$ satisfying (9.2.1) and

$$\| F_{A'} \|_{C^k}(x, t) < \varepsilon$$

on $M \times [0, S' - T'] \cup [S + T, \infty)$. Then A is a standard model of type $(K_1, K_2; C, \varepsilon)$.

Now based on Corollary 9.4 and Proposition 9.3, we can use Uhlenbeck's principle and a standard argument with stratified sets (as in [13]) to establish Theorem 9.5. We put

$$\mathcal{CM}(N; R_j) = \mathcal{M}(N; R_{i_1}) \cup \bigcup_{i_1 < \cdots < i_n} \mathcal{M}(N; R_{i_1}) \times_{R_{i_1}} \cdots \times_{R_{i_n}} \mathcal{M}(M; R_{i_n}, R_j).$$

THEOREM 9.5. *$\mathcal{CM}(N; R_j)$ has a structure of orbifold with corners such that $\mathcal{M}(N; R_j)$ is its interior and $\bigcup_i \mathcal{M}(N; R_i) \times_{R_i} \mathcal{M}(M; R_i, R_j)$ is its codimension one boundary. Furthermore, for each Λ, $\mathcal{M}(N; R_j; \Lambda)$ is relatively compact in $\mathcal{CM}(N; R_j)$.*

We omit the details of proof of Theorem 9.5, since it is essentially the same as the proof of [13, Theorem 7.1], using the results we have already established.
It is easy to see that the proofs of Theorems 2.5 and 3.4 are the same as that of Theorem 9.5. Similar statements are used in Part I (for example in Section 4.) The proof of these also goes in a similar way.
Thus, we established the analytic results we assumed in Part I, except Lemma 4.5. The proof of Lemma 4.5 also goes along the same line. But we need to study also the case when $M = S^2 \times S^1$ and is not a homology sphere. This case is a bit different since we need to deal with reducible connections a bit more seriously.

We consider a complete Riemannian 4-manifold N such that N minus a compact set is isometric to the direct product $S^2 \times S^1 \times (0, \infty)$. (In fact, we consider the case when N may have other ends. But since the boundary condition we put on the other end is the same as the one in previous discussions, we omit mentioning them for a moment.) The set $\bar{R}_+(S^2 \times S^1)$ of the flat connections on $S^2 \times S^1$ is identified with the closed interval $[0, 1]$; here 0 is identified with the trivial connection. We consider a neighborhood of 0. Fix a point p_0 on $S^2 \times S^1$ and put

$$\mathscr{G}_0(S^2 \times S^1) = \{g \in \hat{\mathscr{G}}(S^2 \times S^1) \mid \deg g = 0, g(p_0) = 1\}.$$

For sufficiently large ℓ the intersection $\mathscr{G}_{\ell,0}(S^2 \times S^1) = \mathscr{G}_\ell(S^2 \times S^1) \cap \mathscr{G}_0(S^2 \times S^1)$ is a closed subgroup of $\mathscr{G}_\ell(S^2 \times S^1)$. The quotient of $\tilde{R}(S^2 \times S^1) \subset \mathscr{A}_\ell(S^2 \times S^1)$ (the set of all flat connections on $S^2 \times S^1$) by $\mathscr{G}_{\ell+1}(S^2 \times S^1)$ is diffeomorphic to $SU(2) = S^3$ on which $SU(2) \cong \mathscr{G}_\ell(S^2 \times S^1)/\mathscr{G}_{\ell,0}(S^2 \times S^1)$ acts by conjugation. The inverse image in $SU(2) = S^3$ of a neighborhood, U of $0 \in [0, 1] = \bar{R}_+(S^2 \times S^1)$, is identified with a neighborhood of north pole. This neighborhood is diffeomorphic to D^3, the three-dimensional disk. We fix and choose an $SU(2)$-invariant lift $h: D^3 \to \mathscr{A}_\ell(S^2 \times S^1)$ to the inclusion $h: D^3 \to SU(2) = S^3$. We may assume that $h(0)$ is the zero connection. Now by a simple modification of the proof of Lemma 7.11 we can prove the following.

LEMMA 9.6. *There exists a positive number ε such that if $c \in \operatorname{Im} h$, $a \in \mathscr{A}(S^2 \times S^1)$, $\|c - a\|_{L^2_\ell} < \varepsilon$, then there exists $c' \in \operatorname{Im} h$ and $g \in \mathscr{G}_{\ell,0}(S^2 \times S^1)$, such that*

$$\|c - c'\|_{L^2_\ell} < C\varepsilon \tag{9.6.1}$$

$$c' - g^* a \in (\operatorname{Ker} d_{c'})^\perp. \tag{9.6.2}$$

The proof is omitted since it is identical to the proof of Lemma 7.11. For $x \in D^2$, we fix a connection $A_x \in \mathscr{A}_\ell(N)$ which coincides with $h(x)$ outside a compact subset. We may assume that A_x depends smoothly on x. Now as in Section 2 we define

$$\hat{\mathscr{B}}_{\ell,\delta}(N; U) = \{A \in \Gamma(N, su(2) \otimes \Lambda^1(N)) \mid \exists x \in U \; A_x - A \in \Omega^1_{\ell,\delta}\}.$$

Let $\hat{\mathscr{M}}_{\ell,\delta}(N; U)$ be the set of all ASD connections in $\hat{\mathscr{B}}_{\ell,\delta}(N; U)$, and let $\hat{\mathscr{M}}^*_{\ell,\delta}(N; U)$ be the set of all irreducible connections in $\hat{\mathscr{M}}_{\ell,\delta}(N; U)$. We consider two types of gauge groups acting on them:

$$\mathscr{G}^c_{\ell,\delta}(N) = \{g \in \mathscr{G}(N) \mid \exists u \in \Omega^0_{\ell,\delta} g \exp(-u) = g_{\pm 1}\} \tag{9.7.1}$$

$$\mathscr{G}_{\ell,\delta}(N) = \{g \in \mathscr{G}(N) \mid \exists u \in \Omega^0_{\ell,\delta},\, t \in SU(2),\, g \exp(-u) = g_t\}. \tag{9.7.2}$$

Here g_t is a gauge transform on N which coincides with $t \in SU(2)$ outside a compact subset in N. We put

$$\mathscr{M}_{\ell,\delta}(N; U) \cong \hat{\mathscr{M}}_{\ell,\delta}(N; U)/\mathscr{G}_{\ell,\delta}(N)$$

$$\tilde{\mathscr{M}}_{\ell,\delta}(N; U) \cong \hat{\mathscr{M}}_{\ell,\delta}(N; U)/\mathscr{G}^c_{\ell,\delta}(N).$$

We define $\mathscr{M}^*_{\ell,\delta}(N; U)$ and $\tilde{\mathscr{M}}^*_{\ell,\delta}(N; U)$ in a similar way. Clearly, $SO(3)$ acts on $\tilde{\mathscr{M}}^*_{\ell,\delta}(N; U)$ such that the quotient is $\mathscr{M}^*_{\ell,\delta}(N; U)$. We perform the same construction in a neighborhood of the south pole. The construction in a neighborhood of the other point is simpler. Thus, we construct the diagram as shown in Fig. 7.

We remark here that using Fig. 7 we can easily generalize Theorem 6.1 to our situation. Namely, for generic metric on N the moduli space $\mathscr{M}^*_{\ell,\delta}(N)$, $\tilde{\mathscr{M}}_{\ell,\delta}(N)$ and the map Π^r are smooth.

$$\begin{tikzcd}
\tilde{\mathcal{M}}_{\ell,\delta}(N) \arrow[r] \arrow[d, "/SU(2)"] & SU(2) \arrow[d, "/SU(2)"] \\
\mathcal{M}_{\ell,\delta}(N) \arrow[r] & {[0,1]}
\end{tikzcd}$$

Fig. 7.

To perform Taubes' construction we need to lift the map $\mathcal{M}^*_{\ell,\delta}(N) \to [0, 1]$ locally to $SU(2)$. Namely we take a sufficiently fine open covering $\bigcup_i V_i = \mathcal{M}^*_{\ell,\delta}(N)$. Then, since the principal $SU(2)$-fibration $\tilde{\mathcal{M}}^*_{\ell,\delta}(N) \to \mathcal{M}^*_{\ell,\delta}(N)$ is trivial on V_i it follows that we can construct the lift $V_i \to SU(2)$. We can furthermore lift this map to

$$\varphi_i \colon V_i \to \mathcal{A}_\ell(S^2 \times S^1).$$

Next we take another manifold N' such that N' minus a compact set is isometric to $S^2 \times S^1 \times [0, \infty)$. (We use only the case when $N' \cong S^2 \times D^2$ or $D^3 \times S^1$ in the proof of Lemma 4.5.) We define $\tilde{\mathcal{M}}_{\ell,\delta}(N')$ and Π' is the similar way. By patching N and N' along $S^2 \times S^1$ we obtain a family of manifolds $N(T)$. Now in the same way as Section 8 we can construct maps

$$\mathcal{P}at_{i,T} \colon V_i \times_{SU(2)} \tilde{\mathcal{M}}_{\ell,\delta}(N') \to \mathcal{M}_\ell(N(T))$$

for $T > T_0$ (here we use lifts $V_i \to SU(2)$, $\varphi_i \colon V_i \to \mathcal{A}_\ell(S^2 \times S^1)$). Note that we do not divide the second factor $\tilde{\mathcal{M}}_{\ell,\delta}(N')$ by the $SU(2)$ action. By a dimension counting based on index theorem, we find that the dimensions of $V_i \times_{SU(2)} \tilde{\mathcal{M}}_{\ell,\delta}(N')$ and $\mathcal{M}_\ell(N(T))$ coincides with each other. Now we need the following analogue of Lemma 8.10.

Lemma 9.8. *For each* $v \in \Omega^1_\ell(N(T))$, *and* $w \in T_{[A]}(\tilde{\mathcal{M}}(N))$, $w' \in T_{[A']}(\mathcal{M}(N'))$ *we have*

$$\|\mathcal{P}at_*(w, w') - d_{\mathcal{P}at(A,A')}(u)\|_{\ell;1,2} > \varepsilon \cdot \|\mathcal{P}at_*(w, w')\|_{\ell;1,2}$$

for $T > T_0$. *Here* ε *is a positive number independent of* T.

Proof (sketch). Since the proof is almost the same as the one in [13], we only give an outline of the proof. (In fact, the proof is simpler since one of the connections A is assumed to be irreducible.)

Suppose that the lemma is false. Then there exists a sequence $T_i \to \infty$, u_i, and w, w', A, A' such that $(w, w') \neq (0, 0)$ and that

$$\|u_j\|_{\ell+1;1,2} < C \tag{9.9.1}$$

$$\|\mathcal{P}at_*(w, w') - d_{\mathcal{P}at(A,A')}(u_i)\|_{\ell;1,2} \to 0. \tag{9.9.2}$$

Then we use Eq. (9.9.2) on $S^2 \times S^1 \times [-T_i, T_i]$ and find $s \in su(2)$ such that $|s - u_i|$ converges uniformly to 0 on $S^2 \times S^1 \times \boldsymbol{R}$ by taking a subsequence if necessary. Therefore, by taking a subsequence if necessary, the restriction of u_i to N converges to an element u of

$$\Omega^{0+}_{\ell,\delta}(N) = \{u \,|\, su(2)\text{-valued functions on } N \ \exists s \in su(2) \ \|u - u_s\|_{\ell,\delta} < \infty\}$$

and that $d_A u = w$. Here u_s is an $su(2)$-valued function on N such that u_s coincides with

s outside a compact subset of N. Then the irreducibility of A implies that $u = w = 0$. It, in particular, implies $s = 0$. It follows that the restrictions of u_i to N' converges to an element u' of $\Omega^0_{\ell,\delta}(N')$ such that $d_{A'}u' = w'$. Therefore, by the definition, we have $u' = w' = 0$. This is a contradition.

We can use Lemma 8.10 in a way similar to [13] to prove that the maps $\mathscr{P}a\ell_{i,T}$ are embeddings and its images covers $\mathscr{M}(N(T))$.

Now we are in the position to complete the proof of Lemma 4.5. Let X, ℓ be as in there. Define a metric on $N = X - \ell$ such that N minus a compact set is isometric with $S^2 \times S^1 \times [0, \infty)$. Here we consider the case when N has other end than $S^2 \times S^1 \times [0, \infty)$. We put the boundary condition α there. The definition and the argument go in exactly the same way. Since α is irreducible, we have $\mathscr{M}^*_{\ell,\delta}(N) = \mathscr{M}_{\ell,\delta}(N)$.

Now we first patch N with $D^3 \times S^1$ (then we get X again). In our situation the component of $\tilde{\mathscr{M}}(D^3 \times S^1)$ in question consists of flat connections (one can prove this fact by dimension counting). Hence, $\tilde{\mathscr{M}}(D^3 \times S^1) \cong SU(2)$ and Π^ℓ is the identity. Therefore, we can identify $\mathscr{M}^*_{\ell,\delta}(N) = \mathscr{M}_{\ell,\delta}(N)$ with $\mathscr{M}_{\ell,\delta}(X)$.

Next we patch N with $S^2 \times D^2$. (Then we obtain the manifold X' in Lemma 4.5.) Again $\tilde{\mathscr{M}}(S^2 \times D^2)$ consists of flat connection. In this case, $\tilde{\mathscr{M}}(S^2 \times D^2)$ is a point, which is mapped to the north pole by Π^ℓ. Hence, $\mathscr{M}_{\ell,\delta}(X')$ concides with the inverse image of 0 of $\Pi^r: \mathscr{M}_{\ell,\delta}(N) \to [0, 1]$. By identifying $\mathscr{M}^*_{\ell,\delta}(N) = \mathscr{M}_{\ell,\delta}(N)$ with $\mathscr{M}_{\ell,\delta}(X)$ the map $\Pi^r: \mathscr{M}_{\ell,\delta}(N) \to [0, 1]$ coincides with the holonomy map along our curve ℓ. The proof of Lemma 4.5 is now completed.

Thus we proved all analytic results we assumed in Part I.

May be various arguments of Part II can be generalized to more general situations. It is the author's desire to puruse this line and find a general relation between surgery and moduli spaces of ASD connections. Our Lemma 4.5 is a small step in this direction. But the general problem is still very difficult. Especially for codimension 2 surgery, it seems that one should develop quite different kinds of techniques, which are not yet well understood.

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